Higher Education & Scientific Research Ministry

Al-Mustafa University College

Civil Engineering Department

Numerical Analysis

Third Stage

Numerical Analysis

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Numerical Analysis

RECOMMENDED TEXTBOOK:

Chapra, Steven C., and Canale, Raymond P. (2009). *Numerical Methods for Engineers*. McGraw-Hill, New York.

OTHER REFERENCES:

Al-Khafaji, A. W. and Tooley, J. R. (1986). *Numerical Methods in Engineering Practice*. Holt, Rinehart and Winston, New York.

Gerald, C. F. and Wheatley, P. O. (2004). *Applied Numerical Analysis*. Addison-Wesley, Reading, MA.

Kreyszig, E. (1999). *Advanced Engineering Mathematics*. John Wiley & Sons, New York.

Chapra, Steven .(2011). *Applied Numerical Methods with MATLAB: For Engineers and Scientists*. McGraw-Hill, New York.

Mathews, J. H. and Fink, K. D. (2004). *Numerical Methods using MATLAB*. Pearson Prentice Hall.

Burden, R. L. and Faires, J. D. (2006). *Numerical Analysis*. Brooks/Cole.

Numerical Analysis

INTRODUCTION

Complex engineering problems can be modeled and solved byusing one (or both) of the following approaches.

1. Engineering Analysis (Advanced Mathematics).
2. Numerical Analysis.

The first approach which is the classical one gives **closed-formed solutions**, while the second approach which is the alternative one gives **approximate solutions** (see Figure 1.2). The selection of such an approach depends mainly on the complexity of the problem and accuracy of the results. In fact, problems in modern engineering are so complex that most of them cannot be solved by using the classical approach (*i.e.,* mathematics). So, numerical analysis approach has become one of the most powerful tools to solve many sophisticated problems.

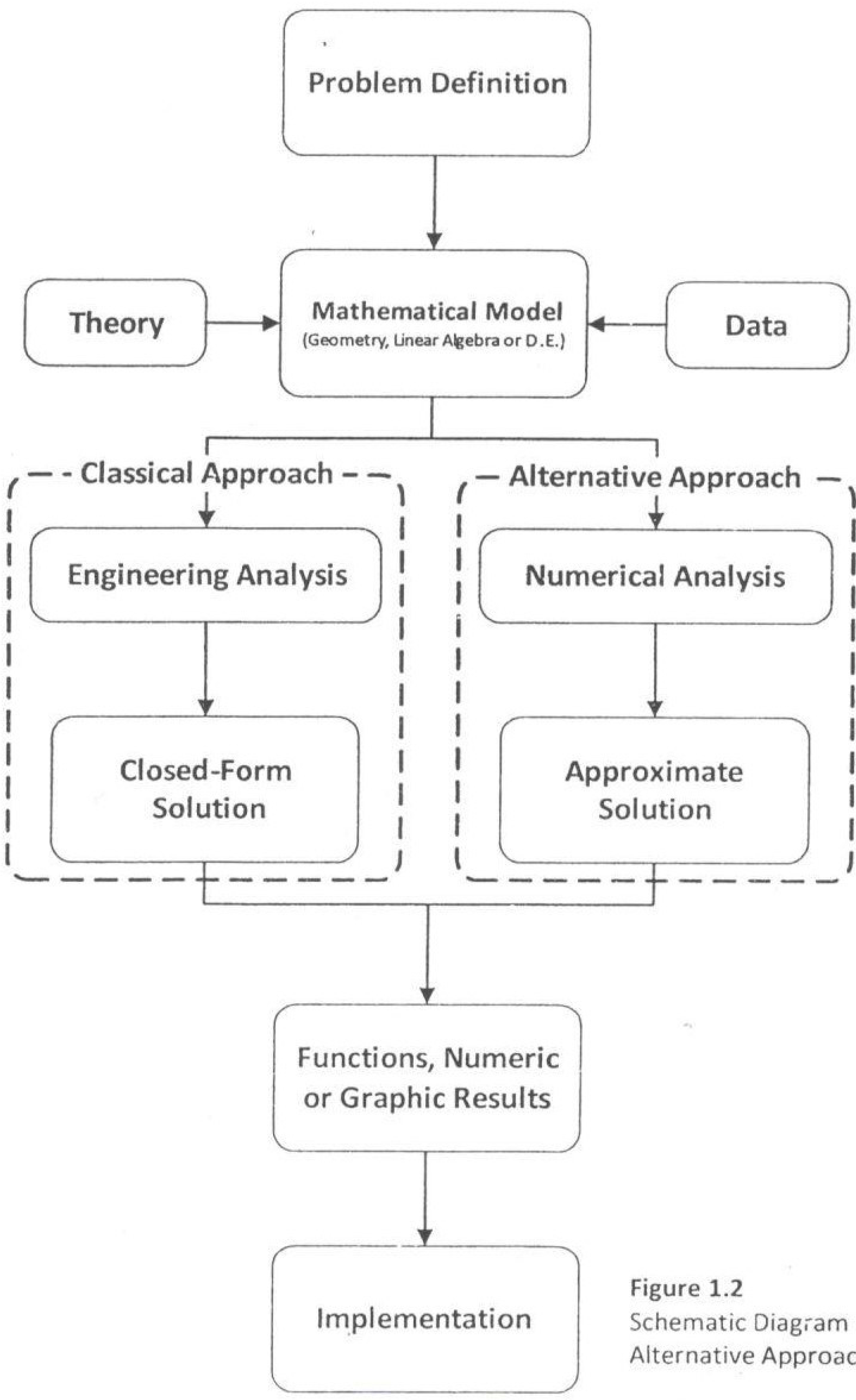

Figure 1.2
Schematic Diagram for the Classical Approach and Alternative Approach in Solving Problems

Numerical Analysis

The Definition of Numerical Analysis [1]

What is numerical analysis? This is more than a philosophical question. A certain wrong answer has taken hold among both outsiders to the field and insiders, distorting the image of a subject at the heart of the mathematical sciences.

Here is the wrong answer:

Numerical analysis is the study of rounding errors.

Of course nobody believes or asserts the above definition quite as baldly as written. But consider the following opening chapter headings from some standard numerical analysis texts:

Isaacson and Keller (1966):	Norms, arithmetic, and well-posed computations.
Hamming (1971):	Roundoff and function evaluation.
Dahlquist and Bjiirck (1974):	Some general principles of numerical calculation How to obtain and estimate accuracy.
Stoer and Bulirsch (1980):	Error analysis.
Conte and de Boor (1980):	Number systems and errors.
Atkinson (1987):	Error: its sources, propagation, and analysis.
Kahaner, Moler and Nash (1989):	Computer arithmetic and computational errors..
Webster's New Collegiate Dictionary (1973):	"The study of quantitative approximations to the solutions of mathematical problems including consideration of the errors and bounds to the errors involved."
Chambers 20th Century Dictionary (1983):	"The study of methods of approximation and their accuracy, etc."
The American Heritage Dictionary (1992):	"The study of approximate solutions to mathematical problems, taking into account the extent of possible errors."
Lloyd N. Trefethen, Department of Computer Science, Cornell University (1992)	Numerical analysis is the study of algorithms for the problems of continuous mathematics

Thus, there is no unique definition to the numerical analysis

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1 Lloyd N. Trefethen, Department of Computer Science, Cornell University

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Numerical Analysis

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Chater 1

APPROXIMATION AND ERRORS

. Errors Definitions

The methods of numerical analysis are **finite processes**, and a numerical result is an **approximate value** of the (unknown) **exact value.** So, numerical errors arise from the use of approximations to represent exact mathematical operation and quantities.

It is worthwhile to mention that many applied engineering problem we cannot obtain analytical solutions (the analytical solution is unknown). Therefore we cannot compute the error associated with numerica' methods exactly.

If $\tilde{a}$ is an approximate value of a quantity whose exact value is a, then the difference

$$|\epsilon| = |\tilde{a} - a| \text{ or } \epsilon = \tilde{a} - a$$

Is called the **absolute error** of $\tilde{a}$ (or the **error** of $\tilde{a}$), hence

$$\tilde{a} = a + \epsilon \Leftrightarrow \text{Approximation=True Value + Error}$$

Also, the relative error ϵ_r of $\tilde{a}$ is defined by

$$\epsilon_r = \frac{|\epsilon|}{a} = \frac{|\tilde{a} - a|}{a} = \frac{\text{Absolute Error}}{\text{True Value}} \qquad a \neq 0$$

If $|\epsilon|$ is much less than $|\tilde{a}|$, then $\tilde{a}$ approaches a and

$$\epsilon_r \approx \frac{|\epsilon|}{\tilde{a}} = \frac{\text{Absolute Error}}{\text{Approximate Value}} \qquad \tilde{a} \neq 0$$

One can also introduce the quantity

$$\gamma = a - \tilde{a} = -\epsilon \quad \text{(The Correction)}$$

thus

$$a = \tilde{a} + \gamma \Leftrightarrow \text{True Value = Approximation + Correction}$$

Finally, an **error bound** for $\tilde{a}$ is a number β such that,

$$|\tilde{a} - a| < \beta, \text{ that is } \epsilon < \beta$$

Remark:

There are many sources of errors. The most important sources are; the accuracy of the mathematical model of the physical situation, the arithmetic system and conditions we use to stop a particular process.

Chater 1

APPROXIMATION AND ERRORS

Round-off and Truncation Errors

The omission of the remaining significant figures is called round-off error

Round-off Error (Rounding)

$\frac{11}{3} \approx 3.666667 + \epsilon \Rightarrow |\epsilon| = 3.333334 \times 10^{-7}$ $(\frac{11}{3} = 3.666666666666666666666666666667)$

$\pi \approx 3.141593 + \epsilon \Rightarrow |\epsilon| = 3.464100 \times 10^{-7}$ $(\pi = 3.14159265358979323846264338327\,95)$

$e \approx 2.718282 + \epsilon \Rightarrow |\epsilon| = 1.715410 \times 10^{-7}$ $(e = 2.71828182845904523536028747135\,27)$

Truncation Error (Chopping)

$\frac{11}{3} \approx 3.666666 + \epsilon \Rightarrow |\epsilon| = 6.666666 \times 10^{-7}$ $(\frac{11}{3} = 3.666666666666666666666666666667)$

$\pi \approx 3.141592 + \epsilon \Rightarrow |\epsilon| = 6.535900 \times 10^{-7}$ $(\pi = 3.14159265358979323846264338327\,95)$

$e \approx 2.718281 + \epsilon \Rightarrow |\epsilon| = 8.284590 \times 10^{-7}$ $(e = 2.71828182845904523536028747135\,27)$

يعتبر موضوع تحليل الأخطاء ذا أهمية بالغة لتحديد أفضل الطرق العددية المستخدمة لحل النماذج الرياضية المختلفة. تعرف الأخطاء من وجهة نظر التحليل العددي بالآتي :

$$\left\{\begin{array}{l}\text{Error}\\ \text{or True Error}\\ \text{or Absolute Error}\end{array}\right\} = \text{True Value} - \text{Approximate Value}$$

(الخطأ أو الخطأ الحقيقي أو الخطأ المطلق) = القيمة الحقيقية − القيمة لتقريبية

وهناك أيضاً ما يعرف :

$$\left\{\begin{array}{l}\text{True Relative Error}\\ \text{or Absolute Relative Error}\end{array}\right\} = \frac{\text{True Value} - \text{Aproximte Value}}{\text{True Value}} \,(100\,\%)$$

(الخطأ النسبي الحقيقي أو الخطأ النسبي الصحيح) = $\frac{\text{القيمة التقريبية} - \text{القيمة الحقيقية}}{\text{القيمة الحقيقية}}$ 100 %

وأيضا هناك :

$$\text{Approximate Relative Error } (\varepsilon_a) = \frac{\text{Present Approximate} - \text{Previous Approximate}}{\text{Present Approximate}} \,(100\,\%)$$

وأن شرط التوقف (Stopping Criterion) بالطرق العددية يكون :

$$\varepsilon_a < \varepsilon_s$$

حيث أن : ε_s يمثل قيمة الخطأ النسبي المرغوب فيه.

Chater 1

APPROXIMATION AND ERRORS

ويمكن حصر مصادر الخطأ في حل المسائل الرياضية على النحو الآتي :

أولا : الخطأ المتأصل Inherent Error :

وهو الخطأ الناتج من قيم البيانات الداخلة والناتجة عن عدم دقة القياسات مثـــل قـــراءات بعض الأجهزة في تجربة مختبرية.

ثانيا : الخطأ التحليلي Analytic Error :

وهو الخطأ الذي ينتج في تحليل تحويل المسالة الى مسالة حسابية، حيـــث أن الإنمـــوذج الرياضي نادرا ما يعطي الصورة الحقيقية للظاهرة كما إننا نضطر أحيانا الى قبول شروط معينة كقواعد لتسهيل المسالة.

ثالثا : الخطأ الحسابي Computational Error :

تظهر الأخطاء الحسابية بصورة عامة عن طريق :

أ) الخطأ المبتور (المقطوع) Truncated Error :

وهو الخطأ الناشئ عن استبدال عمليـــة منتهيـــة (Finite Process) بعمليـــة لانهائيـــة (Infinite Process).

حيث تتضمن بعض المسائل دوال نظرية بشكل سلاسل غير منتهية مثال ذلك :

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

فعند حل مسائل من هذا النوع نضطر الى قطع المتسلسلة عند حد تقريبي يتناسب مع الحل المطلوب، أو نيجاد تكامل الدوال باستخدام بعض صيغ التكامل العددي مثل طريقة شبه المنحرف أو قاعدة سمبسون أو إحلال قيم تقريبية للمشتقات بدل قيم المشتقات نفسها عند حـــل المعـــادلات التفاضلية، وبذلك يتولد الخطأ.

ب) الخطأ المدور (التقريب) Round-Off Error :

ينجم هذا الخطأ عن تقريب الكسور العشرية ذات المراتب العشرية العديدة الى إعداد ذات مراتب عشرية تتناسب مع طبيعة المسالة والدقة المطلوبة مثلا :

$$\frac{1}{3} = 0.333333 = 0.333$$

أو عند تقريب 3.1415926 الى 3.1459 ، أو 0.52357124 الـــى 0.524 ، فخطـــأ التدوير و التقريب الحاصل هو بمثابة الفارق بين العددين.

Type	Result	Example
`short`	Scaled fixed-point format with 5 digits	`3.1416`
`long`	Scaled fixed-point format with 15 digits for double and 7 for single	`3.14159265358979`
`Short e`	Floating-point format with 5 digits	`3.1416e+000`
`long e`	Floating-point format with 15 digits for double and 7 for single	`3.14159265358979e+000`
`short g`	Best of fixed- or floating point format with 5 digits	`3.1416`
`long g`	Best of fixed- or floating point format with 15 digits for double and 7 for single	`3.14159265358979`
`short eng`	Engineering format with at least 5 digits and a power that is multiple of 3	`3.1416e+000`
`Long eng`	Engineering format with at exactly 16 significant digits and a power that is a multiple of 3	`3.14159265358979e+000`

Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

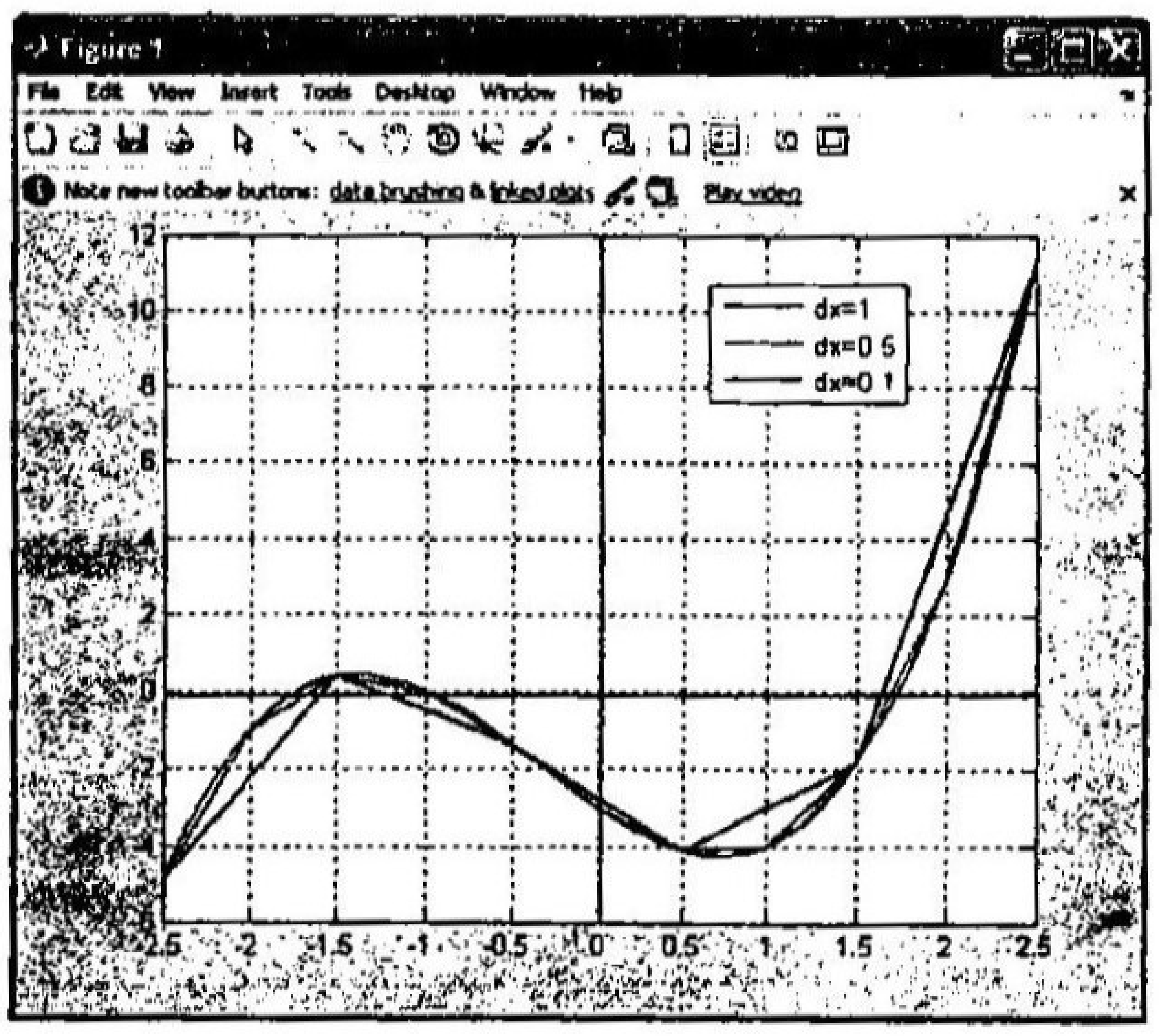
1. Graphical Method

- A simple method for obtaining an estimate of the root of the equation $f(x) = 0$ is to make a plot of the function and observe where it crosses the x axis. This point, which represents the x value for which $f(x) = 0$, provides a rough approximation of the root.

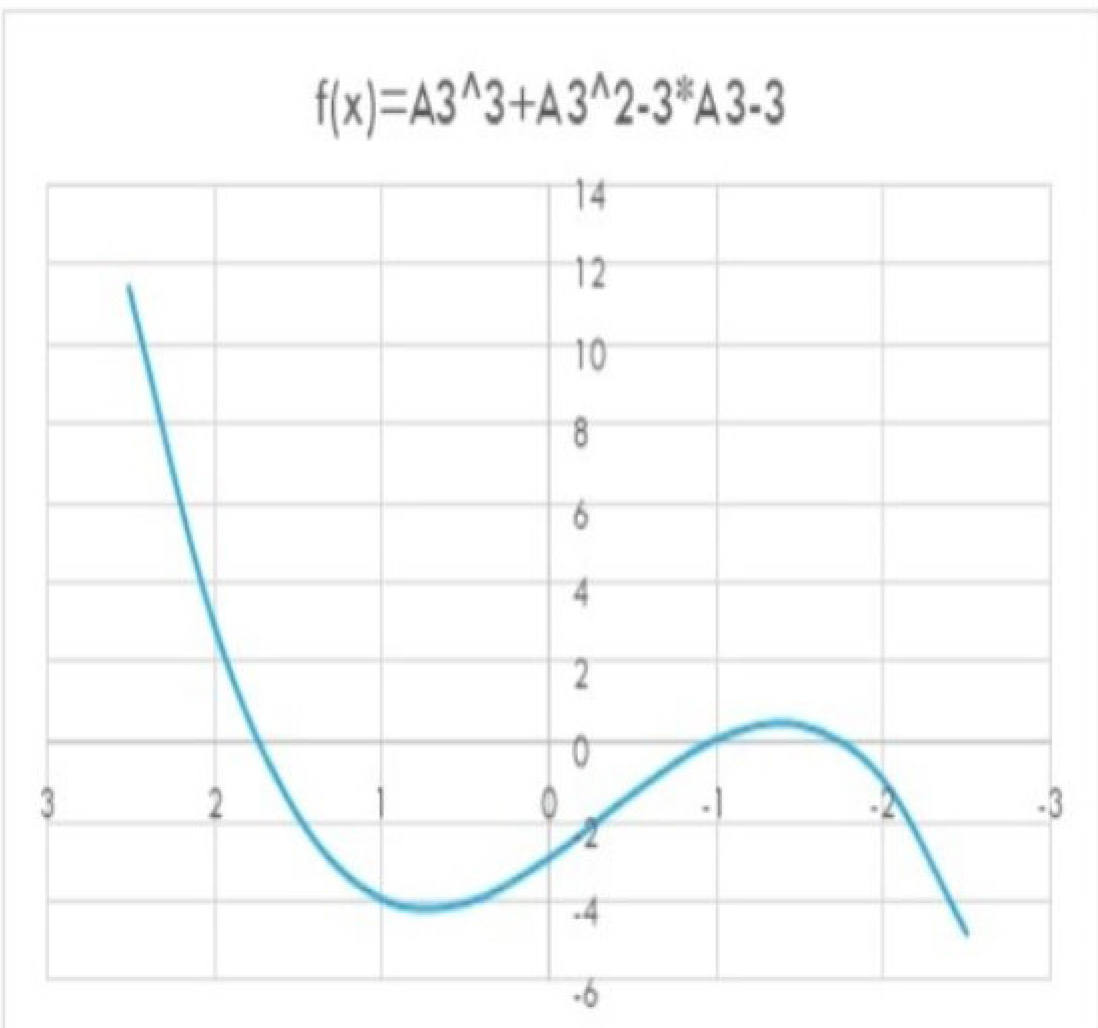
Example 3.1

Find the roots of the equation $f(x) = x^3 + x^2 - 3x - 3$, within the interval $[-2.5, 2.5]$ by using graphical methods (The closed-form solution is $x = -1.732050808, 1.000000000, +1.732050808$).

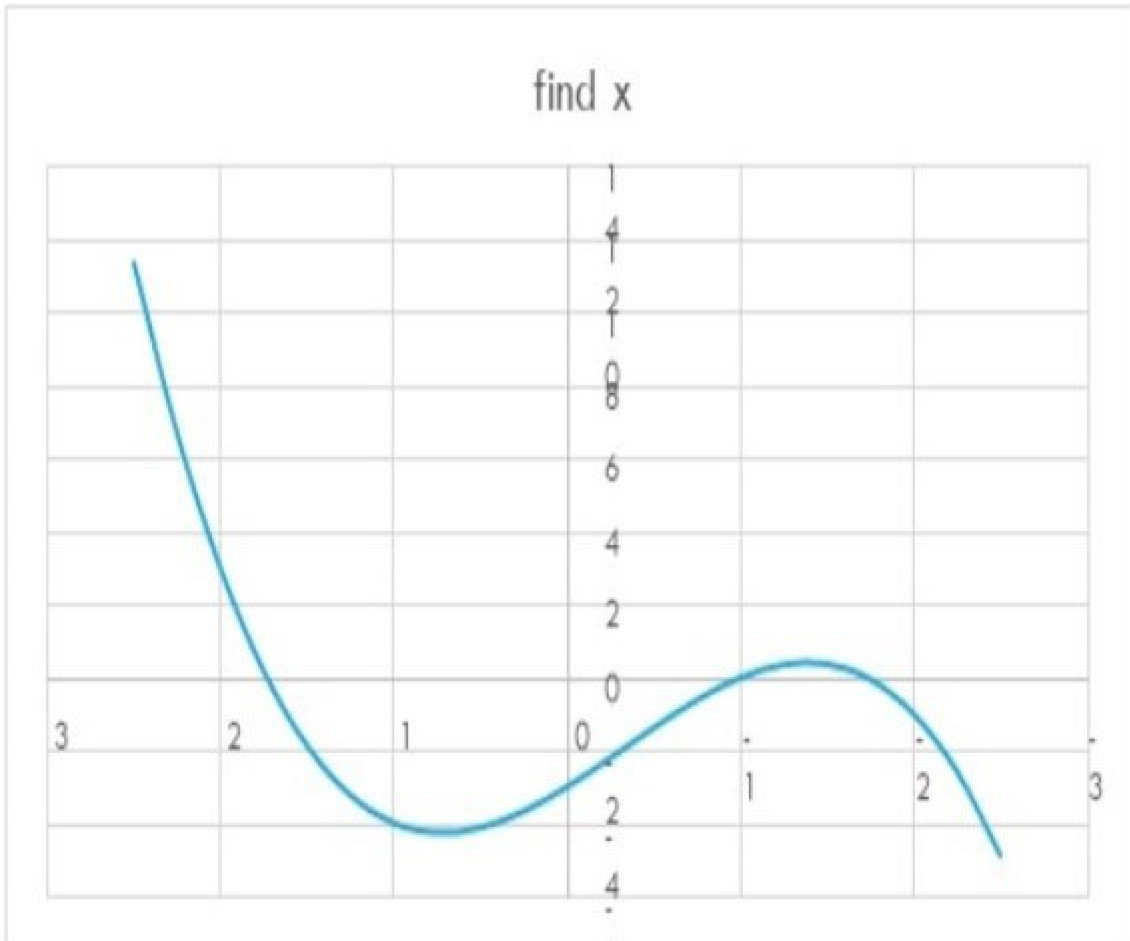
Solution



x	f(x)
-2.5	-4.875
-2	-1
-1.5	0.375
-1	0
-0.5	-1.375
0	-3
0.5	-4.125
1	-4
1.5	-1.875
2	3
2.5	11.375



x	f(x)
-2.5	-4.875
-2.25	-2.57813
-	-
-1.75	-0.04688
-1.5	0.375
-1.25	0.359375
-	0
-0.75	-0.60938
-0.5	-1.375
-0.25	-2.20313
0	-
0.25	-3.67188
0.5	-4.125
0.75	-4.26563
1	-
1.25	-3.23438
1.5	-1.875
1.75	0.171875
•3	
2.25	6.703125
2.5	11.375



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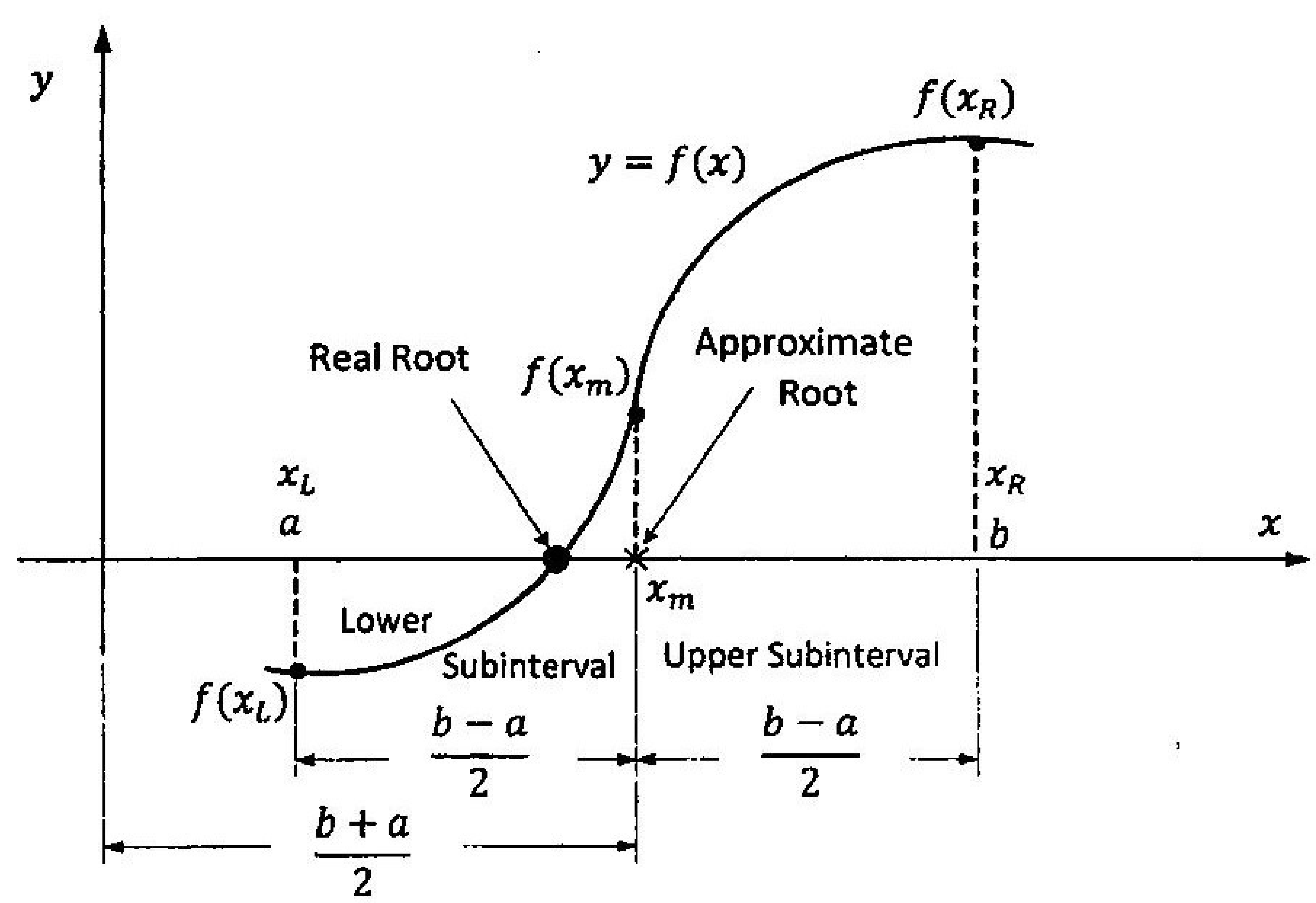
Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

2. Bisection Method (Interval Halving Method or Bracketing Method)

This simple (but slowly convergent method) is based on the intermediate value theorem for continuous functions. If $f(x)$ is real and continuous in the interval from a to b and $f(a)$ and $f(b)$ have opposite signs, that is, $f(a).f(b) < 0$, then there is at least one real root between a and b.

$$x_m = \frac{x_L + x_R}{2}$$



Algorithm: Bisection Method

1. Given a function $f(x)$ real and continuous on an interval $[a, b]$and satisfying $f(a).f(b) < 0$.

   Let $x_L = a$ and $x_R = b$.

2. Estimate the first approximate root

$$x_m = \frac{x_L + x_R}{2}$$

3. If $f(x_m) = 0$, then the root is x_m, accept x_m as a solution and terminate the computation. Else continue.
4. Repeat the following steps until termination:
   a. If $f(x_L).f(x_m) < 0$, the root lies in the lower subinterval. Therefore set $x_R = x_m$. Else (i.e., $f(x_L).f(x_m) > 0$, and the root lies in upper subinterval) set $x_L = x_m$ continue.
   b. Test for termination (Termination Criteria):
      i. If $|x_m^{n+1} - x_m^n| \leq \epsilon$ ($\epsilon > 0$, specified tolerance)
      ii. If $|f(x_m)| \leq \alpha$ ($\alpha > 0$, specified tolerance)
      iii. After N steps (N,fixed)

Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

Example 3.2

Find the root of the equation $f(x) = x^3 + x^2 - 3x - 3$, in the vicinity of 1, by using the Bisection Method. (*Note: Correct to three decimals, 3D*)

Solution

$f(1.000) = -4.000, -ve$
$f(1.500) = -1.875, -ve$
$f(2.000) = +3.000, +ve$
Then $a = 1.500$ and $b = 2.000$, $f(a = 1.5).f(b = 2) < 0$ **O.K.**

$$x_m = \frac{1}{2}(x_L + x_R)$$

n	x_L	x_R	$f(x_L)$	$f(x_R)$	x_m	$f(x_m)$
1	1.500	2.000	−1.875	3.000	1.750	0.172
2	1.500	1.750	−1.875	0.172	1.625	−0.943
3	1.625	1.750	−0.943	0.172	1.688	−0.409
4	1.688	1.750	−0.409	0.172	1.719	−0.125
5	1.719	1.750	−0.125	0.172	1.734	0.022
6	1.719	1.734	−0.125	0.022	1.727	−0.052
7	1.727	1.734	−0.052	0.022	1.731	−0.015
8	1.731	1.734	−0.015	0.022	1.732	0.004
9	1.731	1.732	−0.015	0.004	1.731	−0.006
10	1.731	1.732	−0.006	0.004	1.732	−0.001
11	1.732	1.732	−0.001	0.004	1.732	0.001

The root is $\mathbf{1.732}$ because $|x_m^{11} - x_m^{10}| = 0$ <u>**Ans.**</u>

Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

Example 3.3

Find the Intersection point between the function $y_1 = x^3$ and $y_2 = 1 - 3x$ by using the Bisection Method within the interval $[-1, 1.5]$. (*Note: Correct to 6D and use* $\epsilon = 0.0006$).

Solution

Let $y_1 = y_2 \leftrightarrow x^3 = 1 - 3x$

$\therefore f(x) = x^3 + 3x - 1 = 0,\ \ a = -1, \quad b = 1.5.$

$f(-1) = -5$ *and* $f(1.5) = 6.875$ *and satisfying* $f(a).f(b) < 0\ \ O.K.$

Let $x_L = a$ *and* $x_R = b$ (*Note: One can assume* $a = 0$ *because* $f(a = 0).f(b = 1.5) < 0$)

$$x_m = \frac{1}{2}(x_L + x_R)$$

n	x_L	x_R	$f(x_L)$	$f(x_R)$	x_m	$f(x_m)$	$\lvert x_m^{n+1} - x_m^n \rvert$
1	−1.000000	1.500000	−5.000000	6.875000	0.250000	−0.234375	−
2	0.250000	1.500000	−0.234375	6.875000	0.875000	2.294922	0.625
3	0.250000	0.875000	−0.234375	2.294922	0.562500	0.865479	0.3125
4	0.250000	0.562500	−0.234375	0.865479	0.406250	0.285797	0.15625
5	0.250000	0.406250	−0.234375	0.285797	0.328125	0.019703	0.07813
6	0.250000	0.328125	−0.234375	0.019703	0.289063	−0.108659	0.03906
⋮	⋮	⋮	⋮	⋮	⋮	⋮	
12	0.322021	0.323242	−0.000543	0.003501	0.322632	0.001479	0.00061
13	0.322021	0.322632	−0.000543	0.001479	0.322327	0.000468	0.00031

Then the root is $\mathbf{0.322327}$ because $\lvert x_m^{13} - x_m^{12} \rvert = 0.00031 < \epsilon$,

and the intersection point is ($x = \mathbf{0.322327}, y = \mathbf{0.033488}$) <u>Ans.</u>

Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

Example 3.4

Estimate the local maximum point of the equation $y = 3 - \frac{1}{2}\cos 2x, [0.1\pi, 0.8\pi]$ by using the Bisection Method. (*Note: Correct to 3D*).

Solution

Let $f(x) = y' = \sin 2x$

$\therefore$ *assume* $a = 0.10\pi = 0.314$ *and* $b = 0.8\pi = 2.513.$

$f(a).f(b) < 0$ **O.K.**

Let $x_L = a$ *and* $x_R = b$

$$x_m = \frac{1}{2}(x_L + x_R)$$

n	x_L	x_R	$f(x_L)$	$f(x_R)$	x_m	$f(x_m)$
1	0.314	2.513	0.588	-0.951	1.414	0.309
2	1.414	2.513	0.309	-0.951	1.963	-0.707
3	1.414	1.963	0.309	-0.707	1.689	-0.233
4	1.414	1.689	0.309	-0.233	1.551	0.039
5	1.551	1.689	0.039	-0.233	1.620	-0.098
6	1.551	1.620	0.039	-0.098	1.586	-0.029
⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	1.570	1.572	0.001	-0.002	1.571	0.000

Then the root is $\mathbf{1.571}$ because $|f(x_m^{12})| = 0,$

and the local maximum point is $(x = \mathbf{1.571}, y(1.571) = \mathbf{3.500})$ **Ans.**

Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

Example 3.5

Determine the roots of the quadratic equation $y = x^2 + x - 1 = 0, [-2,2]$ by using the Bisection Method. (*Note: Correct to 6D and* $N_{max} = 5$).

Solution

Assume that there are two roots (positive and negative) to be checked later.

Let $f(x) = y$

$$x_m = \frac{1}{2}(x_L + x_R)$$

For positive root, assume $a = 0$ *and* $b = 2 \rightarrow f(a).f(b) < 0$ **O.K.**

n	x_L	x_R	$f(x_L)$	$f(x_R)$	x_m	$f(x_m)$
1	0.000000	2.000000	-1.000000	5.000000	1.000000	1.000000
2	0.000000	1.000000	-1.000000	1.000000	0.500000	-0.250000
3	0.500000	1.000000	-0.250000	1.000000	0.750000	0.312500
4	0.500000	0.750000	-0.250000	0.312500	0.625000	0.015625
5	0.500000	0.625000	-0.250000	0.015625	0.562500	-0.121094

Then the positive root is **0.562500** because $N_{max} = 5$ *but* $\alpha = -0.121094$ **Ans.**

For negative root, assume $a = -2$ *and* $b = 0 \rightarrow f(a).f(b) < 0$ **O.K.**

n	x_L	x_R	$f(x_L)$	$f(x_R)$	x_m	$f(x_m)$
1	-2.000000	0.000000	1.000000	-1.000000	-1.000000	-1.000000
2	-2.000000	-1.000000	1.000000	-1.000000	-1.500000	-0.250000
3	-2.000000	-1.500000	1.000000	-0.250000	-1.750000	0.312500
4	-1.750000	-1.500000	0.312500	-0.250000	-1.625000	0.015625
5	-1.625000	-1.500000	0.015625	-0.250000	-1.562500	-0.121094

Then the negative root is **−1.562500** because $N_{max} = 5$ *but* $\alpha = -0.121094$ **Ans.**

Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

3. Newton's Method (Newton-Raphson Method or Open Method)

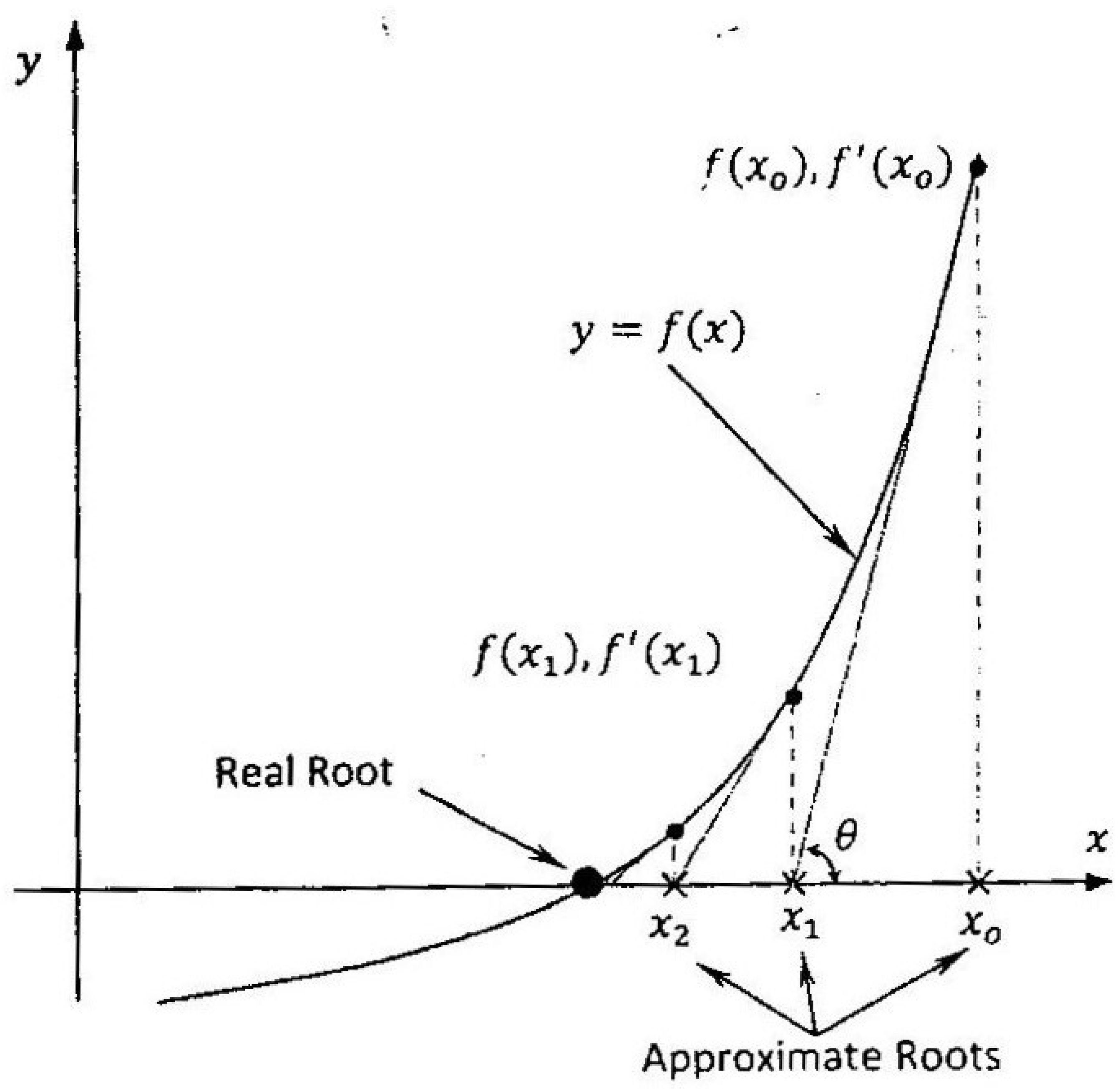
This method is commonly used because of its simplicity and great speed. If the initial guess at the root is x_o, a tangent can be extended from the point $(x_o, f(x_o))$. The point where this tangent crosses the x axis usually represents an improved estimate of the root. This method can be derived geometrically as follows (see the figure below)

$$\tan\theta = f'(x_o) = \frac{f(x_o) - 0}{x_o - x_1}$$

$$x_o - x_1 = \frac{f(x)}{f'(x)} \quad \Rightarrow \quad x_1 = x_o - \frac{f(x_o)}{f'(x_o)}$$

Or generally,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0$$



Algorithm: Newton's Method

1. Given a function $f(x)$ real and continuous and has a continuous derivative.
2. Given a starting value x_o(initial guess).
3. Repeat the following steps until termination:
   a. Compute $f(x_n), f'(x_n)$ (if $f'(x_n) = 0$ stop, pitfall).
   b. If $f(x_n) = 0$, then the root is x_n and terminate the computation. Else,
   c. Compute
   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$
   d. Test for termination (Termination Criteria):
      i. If $|x_m^{n+1} - x_m^n| \leq \epsilon$ ($\epsilon > 0$, specified tolerance)
      ii. If $|f(x_m)| \leq \alpha$ ($\alpha > 0$, specified tolerance)
      iii. After N steps (N, fixed)

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Example 3.6

Find the positive solution of $2\sin x = x$ by using Newton's Method. (*Assume* $x_0 = 2.000$ *and correct to three decimals,* 3*D*)

Solution

$f(x) = x - 2\sin x = 0$
$f'(x) = 1 - 2\cos x$

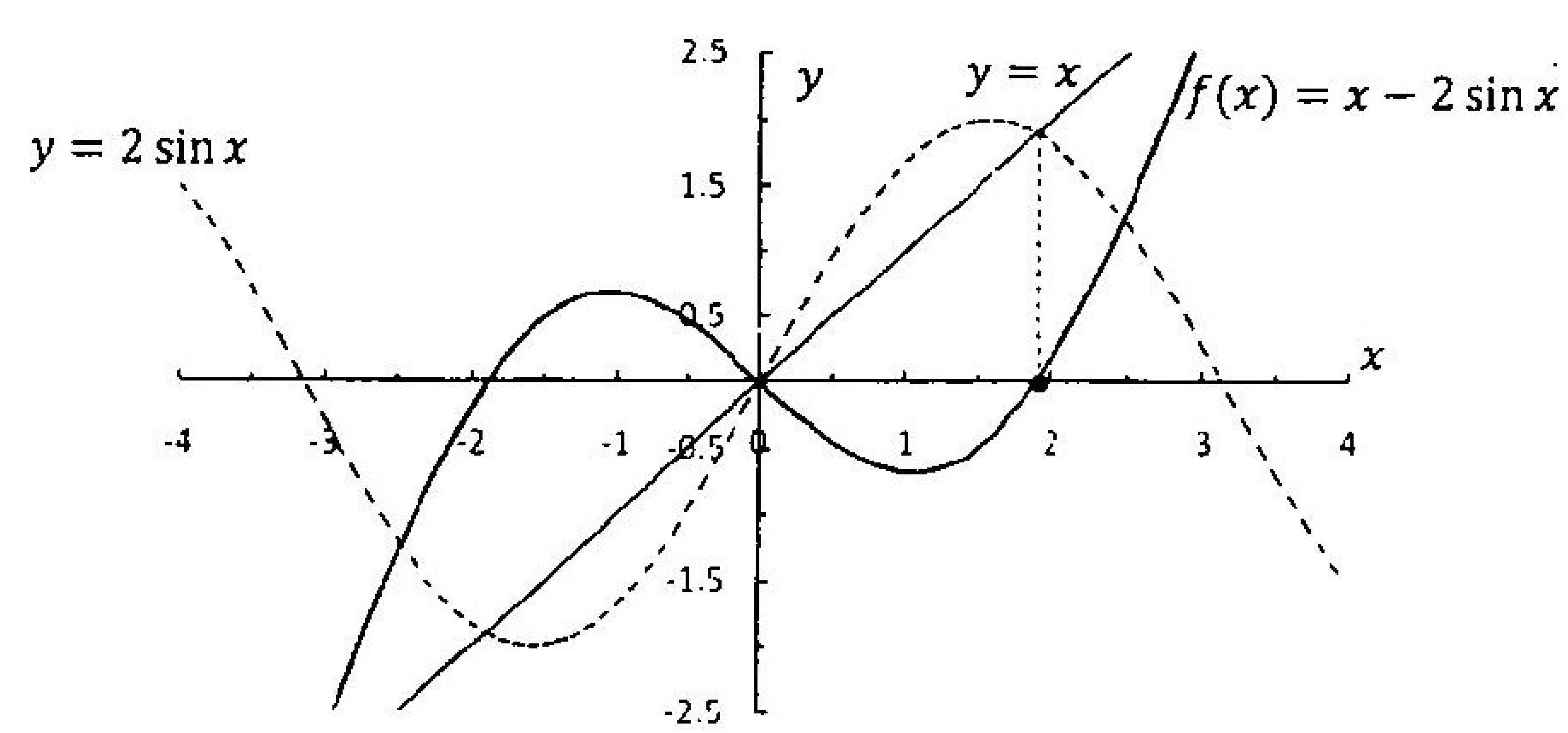
$$x_{n+1} = x_n - \frac{x_n - 2\sin x_n}{1 - 2\cos x_n},$$

n	x_n	x_{n+1}	$\lvert f(x_{n+1})\rvert$	$x_{n+1} - x_n$
0	2.000	1.901	0.009	0.099
1	1.901	1.896	0.000	0.005
2	1.896	1.895	0.000	0.001
3	1.895	1.895	0.000	0.000

The root is **1.895** because $|x_m^3 - x_m^4| = 0$ *and* $f(x_m^4) = 0$ **Ans.**

(In this example both $f(1.896) = 0$ and $f(1.895) = 0$ but the more accurate solution is $x =$ because if one uses 5*D* instead of 3*D* the two answers will be $x_1 = 1.89551$ and $x_2 = 1.89549$ and 4*D* will give $x_1 = x_2 = 1.8955$).

It is worthwhile to mention that the calculation in the above table was performed by a computer.



Example 3.7 (Square Root) :- Use Newton's Method to find the solution of $x = \sqrt{c}$ where c is any positive number.

Solution: Let $x^2 = c \Rightarrow f(x) = x^2 - c \rightarrow f'(x) = 2x$

$$x_{n+1} = x_n - \frac{x_n^2 - c}{2x_n} = x_n - \frac{1}{2}\left(x_n - \frac{c}{x_n}\right) = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

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Example 3.8

Use the solution of Example 3.7 to find $\sqrt{2}$. (*Assume* $x_o = 1$ *and correct to* $6D$)

Solution

Let $C = 2$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

n	x_{n+1}	$\lvert f(x_{n+1})\rvert$	$x_{n+1} - x_n$
0	1.500000	0.250000	0.500000
1	1.416667	0.006944	0.083333
2	1.414216	0.000006	0.002451
3	1.414214	0.000000	0.000002
4	1.414214	0.000000	0.000000

Then the root is **1.414214** because $\lvert f(x_3)\rvert = 0$ *and* $\lvert x_m^3 - x_m^4\rvert = 0,$

Example 3.9

Estimate the intersection points of the functions $y_1 = -0.4x^2$ and $y_2 = 5\sin x$ by using Newton's Method. (Use $-8, -3, 1, 2$ and 8 as starting points, correct to six decimals).

Solution

$f(x) = 0.4x^2 + 5\sin x$ (Let $y_1 = y_2$)

$f'(x) = 0.8x + 5\cos x$

$$x_{n+1} = x_n - \frac{0.4x_n^2 + 5\sin x_n}{0.8x_n + 5\cos x_n},$$

n	$x_o = -8$		$x_o = -3$		$x_o = 1$		$x_o = 2$		$x_o = 8$	
	x_{n+1}	$\lvert f(x_{n+1})\rvert$	x_{n+1}	$\lvert f(x_{n+1})\rvert$	x_{n+1}	$\lvert f(x_{n+1})\rvert$	x_{n+1}	$\lvert f(x_{n+1})\rvert$	x_{n+1}	$\lvert f(x_{n+1})\rvert$
0	−5.10232	15.03815	−2.606202	0.166031	−0.315819	−1.513078	14.78563	91.43097	2.614933	5.248392
1	1.792022	6.162683	−2.580200	0.001145	0.020416	0.102242	4.406014	2.998018	4.967936	5.034528
2	−16.5226	112.8365	<u>−2.580018</u>	0.000000	0.000030	0.000152	2.919484	4.510789	4.006822	2.615629
3	−9.74512	39.56136	<u>−2.580018</u>	0.000000	<u>0.000000</u>	0.000000	4.694275	3.815309	74.94515	2244.5217(
4	−6.59074	15.8615			<u>0.000000</u>	0.000000	3.653223	2.890418	40.12027	647.15287(
5	24.68136	241.4867					5.664426	9.934165	17.28385	114.49258;
6	14.7208	90.85314					4.509899	3.237832	9.018727	34.509897
7	4.649881	3.658321					3.265716	3.646935	−4.14524	11.090373
8	3.576293	3.010256					4.818291	4.314383	−2.29757	−1.625104
9	5.374591	7.611287					3.833981	2.687877	−2.61249	0.206227
10	4.342404	2.880925					7.273655	25.34385	−2.5803	0.001746
⋮	⋮	⋮					⋮	⋮	−2.58002	0.000000
⋮	$n = 43$ <u>0.000000</u>	$n = 43$ 0.000000					$n = 53$ <u>−2.580018</u>	$n = 53$ 0.000000	−2.58002	0.000000
n	$n = 44$ <u>0.000000</u>	$n = 44$ 0.000000					$n = 54$ <u>−2.580018</u>	$n = 54$ 0.000000		

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Example 3.10

The vertical stress increment due to an infinite strip load is shown as follows

$$\Delta\sigma_z = \frac{P}{\pi}[\alpha + \sin\alpha\cos(\alpha + 2\delta)]$$

Determine the value of the angle α if $P = 100\ lb/ft$, $\delta = \pi/4$ and $\Delta\sigma_z = 10$ lb/ft2, using the Newton's Method (*Note: Correct to four decimals*).

Solution

Substitute P=100 lb/ft, δ=π/4 and $\Delta\sigma_z$ =10 lb/ft^2 into the stress equation as follows

$$10 = \frac{100}{\pi}\left[\alpha + \sin\alpha\cos\left(\alpha + \frac{2\pi}{4}\right)\right] \Rightarrow 10\pi = 100\left[\alpha + \sin\alpha\cos\left(\alpha + \frac{\pi}{2}\right)\right]$$

and

$\cos(\alpha + \pi/2) = -\sin\alpha$ then

$10\pi = 100[\alpha + \sin\alpha(-\sin\alpha)] \Rightarrow 10\pi = 100[\alpha - \sin^2\alpha]$

or

$f(\alpha) = 100[\alpha - \sin^2\alpha] - 10\pi = 0$ (the equation)

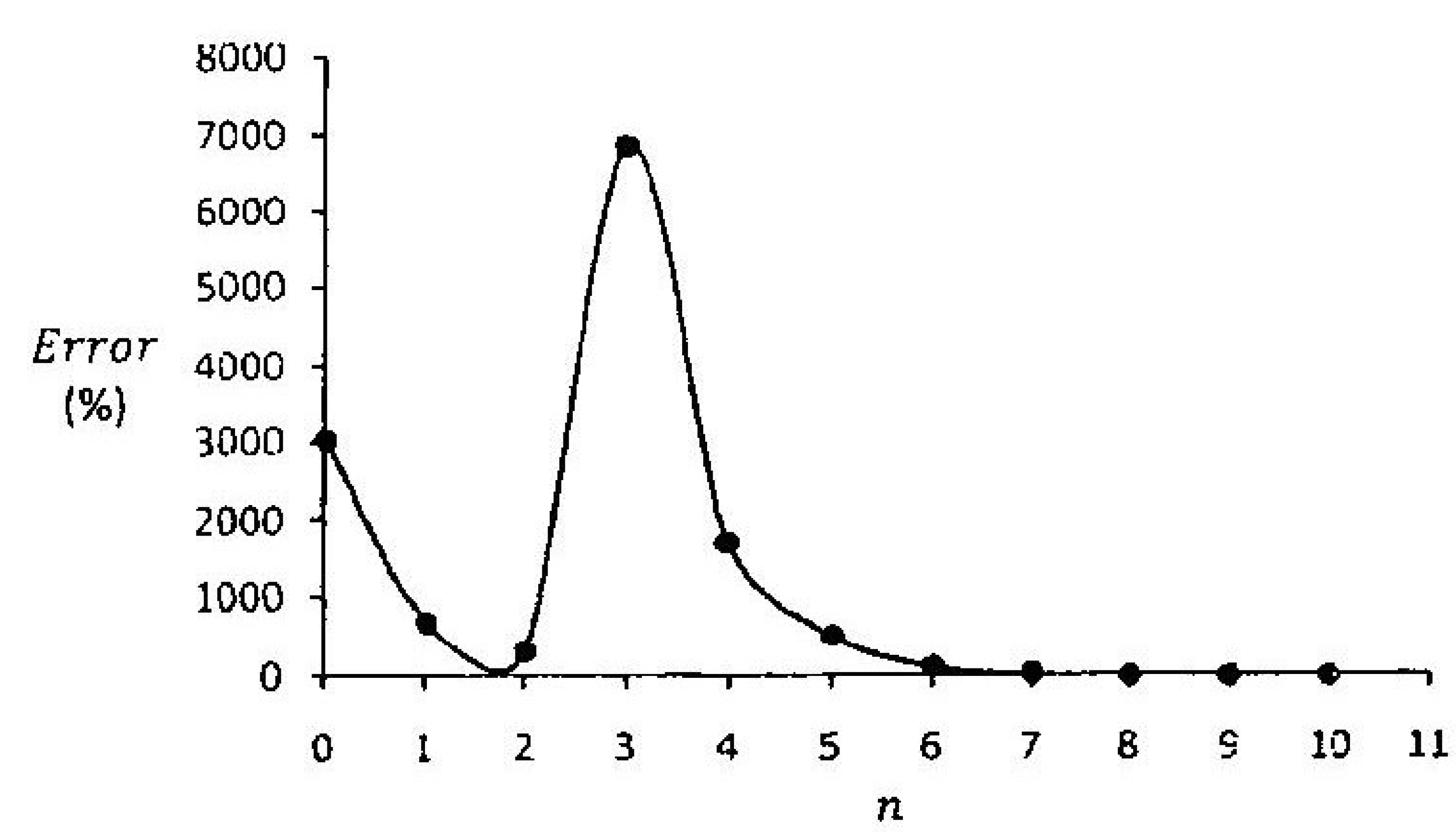
$f'(\alpha) = 100[1 - 2\sin\alpha\cos\alpha]$ (the derivative)

then

$$\alpha_{n+1} = \alpha_n - \frac{100[\alpha_n - \sin^2\alpha_n] - 10\pi}{100[1 - 2\sin\alpha_n\cos\alpha_n]}$$

Let $\alpha_o = 0.0000$

n	α_{n+1}	$Error = \lvert f(\alpha_{n+1})\rvert$
0	0.3142	−9.5492
1	0.5458	−3.7825
2	0.8817	−2.8167
3	2.4059	164.1389
4	1.5832	26.9180
5	1.3205	6.7699
6	1.1903	1.4078
7	1.1450	0.1446
8	1.1392	0.0022
9	1.1391	0.0000



The value of the angle α is 1.1391 because $|f(\alpha_9)| = 0$ ***Ans.***

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4. False Position Method (Regula Falsi or Linear Interpolation Method)

In this method one can approximate the curve of $f(x)$ by a chord for solving $f(x) = 0$ as shown in the figure below

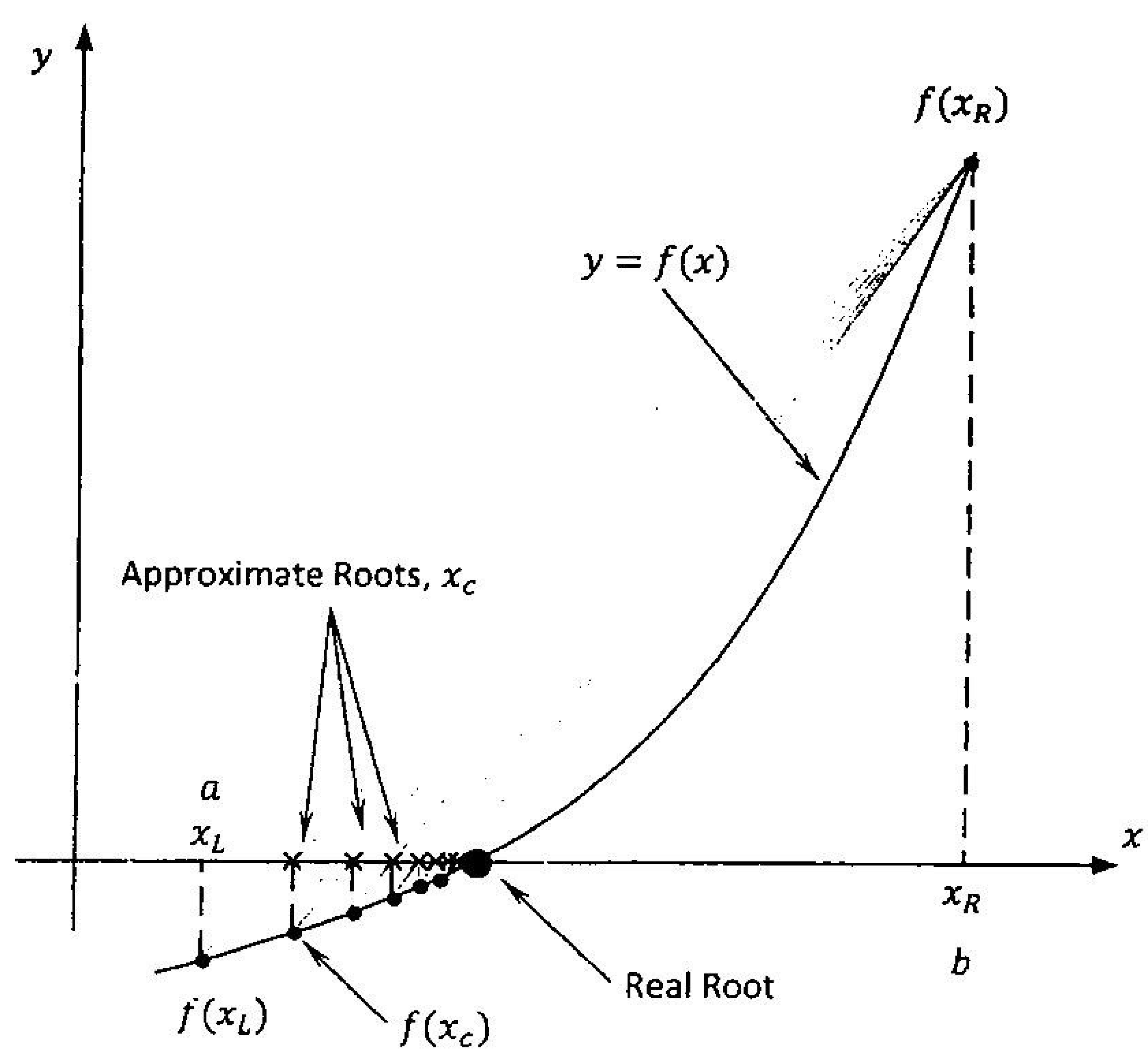
$$\frac{f(x_R) - f(x_L)}{x_R - x_L} = \frac{f(x_R) - 0}{x_R - x_c} \Rightarrow (f(x_R) - f(x_L))(x_R - x_c) = f(x_R)(x_R - x_L)$$

$$x_R f(x_R) - x_c f(x_R) - x_R f(x_L) + x_c f(x_L) = x_R f(x_R) - x_L f(x_R)$$

$$x_c[f(x_L) - f(x_R)] = x_R f(x_L) - x_L f(x_R)$$

Or

$$x_c = \frac{x_L\, f(x_R) - x_R\, f(x_L)}{f(x_R) - f(x_L)}$$



Algorithm: False Position Method

1. Given a function $f(x)$ real and continuous on an interval $[a, b]$ and satisfying $f(a).f(b) < 0$.

   Let $x_L = a$ and $x_R = b$.

2. Estimate the first approximate root

$$x_c = \frac{x_L\, f(x_R) - x_R\, f(x_L)}{f(x_R) - f(x_L)}$$

3. If $f(x_m) = 0$, then the root is x_m, accept x_m as a solution and terminate the computation. Else continue.

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4. Repeat the following steps until termination:
   a. If $f(x_L).f(x_m) < 0$, the root lies in the lower subinterval. Therefore set $x_R = x_m$. Else (i.e., $f(x_L).f(x_m) > 0$, and the root lies in upper subinterval) set $x_L = x_m$ continue.
   b. Test for termination (Termination Criteria):
      i. If $|x_m^{n+1} - x_m^n| \leq \epsilon$ ($\epsilon > 0$, specified tolerance)
      ii. if $|f(x_m)| \leq \alpha$ ($\alpha > 0$, specified tolerance)
      iii. After N steps (N,fixed)

Example 3.11
Find the local maximum point of the function $y = 1 - \cos x - e^x, [-5,0]$ by using the False Position Method. (*Note: Correct to 4D*).

Solution
Let $f(x) = y' = \sin x - e^x$
$\therefore$ *assume* $a = -5$ *and* $b = 0$.
$f(a).f(b) < 0\ O.K.$
Let $x_L = a$ *and* $x_R = b$

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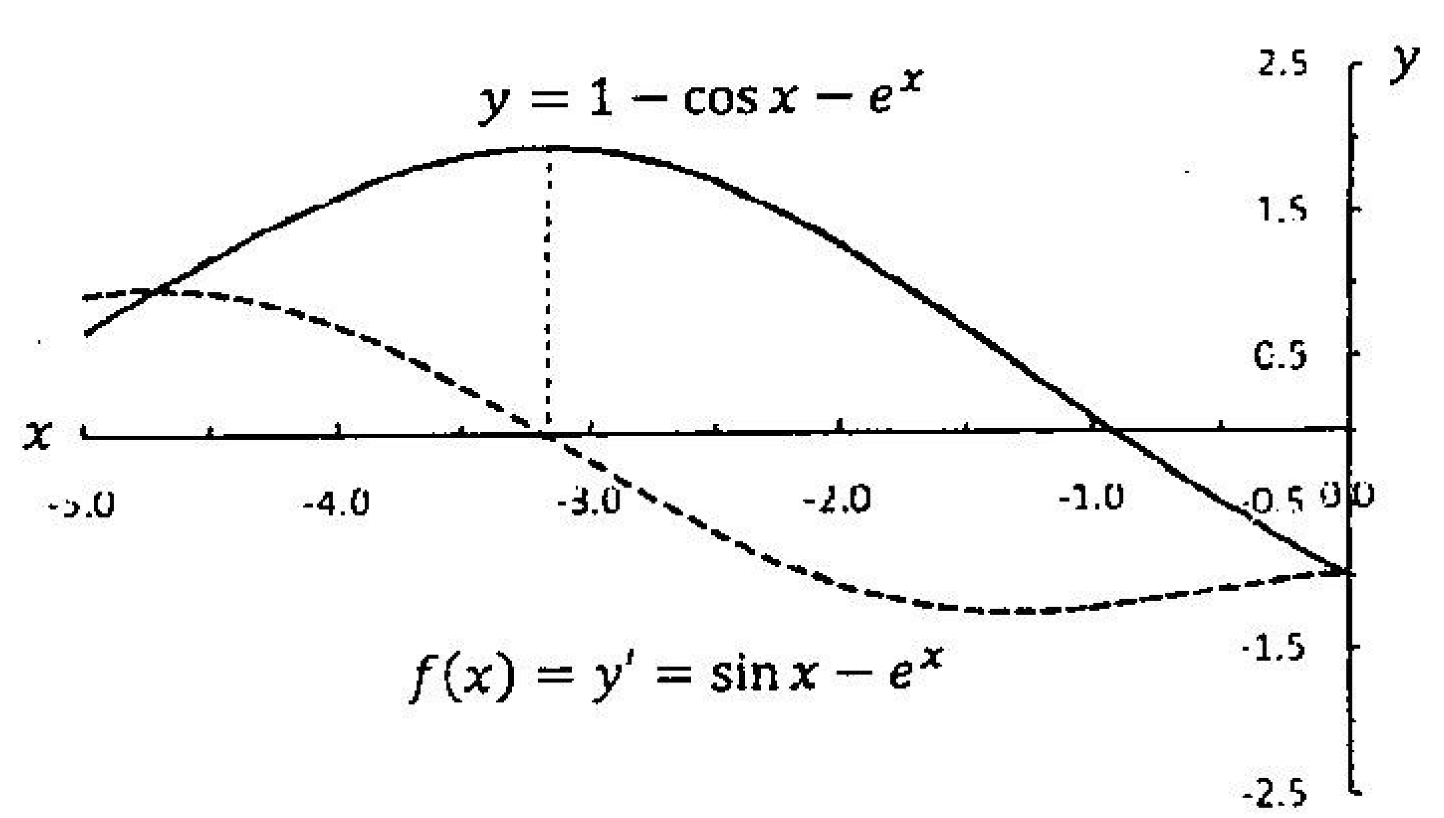
$$x_c = \frac{x_L\, f(x_R) - x_R\, f(x_L)}{f(x_R) - f(x_L)}$$

n	x_L	x_R	$f(x_L)$	$f(x_R)$	x_c	$f(x_c)$
1	−5.0000	0.0000	0.9522	−1.0000	−2.5612	−0.6255
2	−5.0000	−2.5612	0.9522	−0.6255	−3.5282	0.3476
3	−3.5282	−2.5612	0.3476	−0.6255	−3.1827	−0.0003
4	−3.5282	−3.1827	0.3476	−0.0003	−3.1831	0.0000

Then the root is $-\mathbf{3.1831}$ because $|f(x_4)| = 0.0000$,
and the local maximum point is $(x = -\mathbf{3.1831}, y(-3.1831) = \mathbf{1.9577})$ ***Ans***.



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Example 3.12

Estimate the negative root of the equation $f(x) = x^3 - 3x + 1$, $[-2, 1.5]$, using the False-Position Method. (*Note: Use six decimals and* $\epsilon = 0.000008$)

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Solution

Let $a = -2$ and $b = 0$ (negative root)

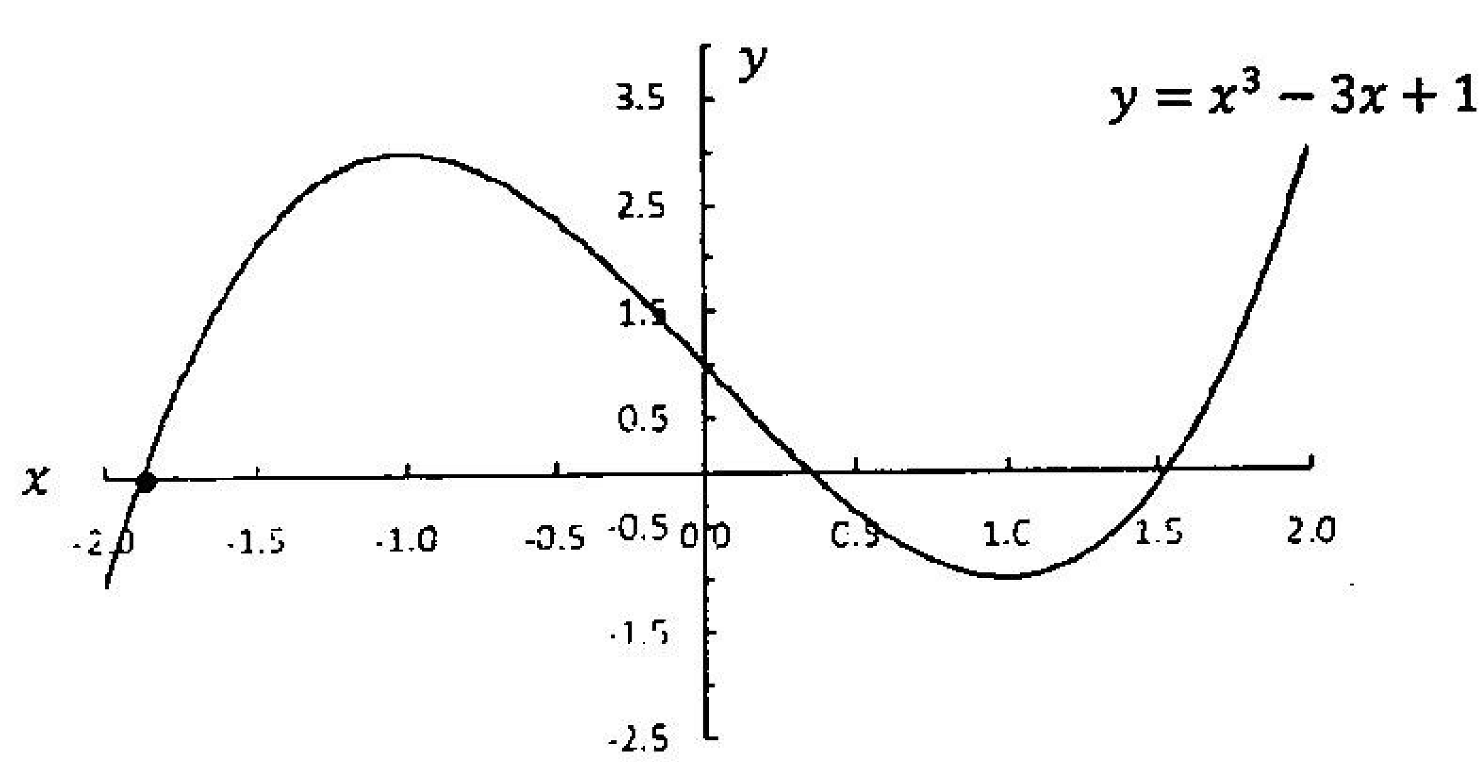
$f(a).f(b) < 0$ O.K.

Let $x_L = a$ *and* $x_R = b$

$$x_c = \frac{x_L\, f(x_R) - x_R\, f(x_L)}{f(x_R) - f(x_L)}$$

n	x_L	x_R	$f(x_L)$	$f(x_R)$	x_c	$f(x_c)$	$\lvert x_{n+1} - x_n \rvert$
1	−2.000000	0.000000	−1.000000	1.000000	−1.000000	3.000000	−
2	−2.000000	−1.000000	−1.000000	3.000000	−1.750000	0.890625	0.750000
3	−2.000000	−1.750000	−1.000000	0.890625	−1.867769	0.087484	0.117769
4	−2.000000	−1.867769	−1.000000	0.087484	−1.878406	0.007432	0.010638
5	−2.000000	−1.878406	−1.000000	0.007432	−1.879303	0.000623	0.000897
6	−2.000000	−1.879303	−1.000000	0.000623	−1.879378	0.000052	0.000075
7	−2.000000	−1.879378	−1.000000	0.000052	−1.879385	0.000004	0.000006

The $-ve$ root is $\mathbf{-1.879385}$ because $|x_7 - x_6| < \epsilon$ ***Ans.***



Example 3.13

Find the local max./min. points of the equation $y = \frac{x^4}{4} - \frac{3x^2}{2} + x - 1$, within the interval $[-3, 1.5]$, using the False-Position Method. (*Note: Use six decimals and* $\alpha = 0.000005$)

Solution

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$$y = \frac{x^4}{4} - \frac{3x^2}{2} + x - 1$$

$f(x) = y' = x^3 - 3x + 1 = 0$

Let $a = 0$ and $b = 1.5$ (for positive root)

$f(a).f(b) < 0$ O.K.

Let $x_L = a$ *and* $x_R = b$

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$$x_c = \frac{x_L\, f(x_R) - x_R\, f(x_L)}{f(x_R) - f(x_L)}$$

n	x_L	x_R	$f(x_L)$	$f(x_R)$	x_c	$\lvert f(x_c)\rvert$
1	0.000000	1.500000	1.000000	−0.125000	1.333333	0.629630
2	0.000000	1.333333	1.000000	−0.629630	0.818182	0.906837
3	0.000000	0.818182	1.000000	−0.906837	0.429078	0.208237
4	0.000000	0.429078	1.000000	−0.208237	0.355127	0.020595
5	0.000000	0.355127	1.000000	−0.020595	0.347961	0.001753
6	0.000000	0.347961	1.000000	−0.001753	0.347352	0.000147
7	0.000000	0.347352	1.000000	−0.000147	0.347301	0.000012
8	0.000000	0.347301	1.000000	−0.000012	0.347297	0.000001

The $+ve$ root is 0.347297 because $|f(x_c^8)| = 0.000001 < \alpha = 0.000005$
The maximum point is $x = \mathbf{0.347297}$ and $y = f(0.347297) = \mathbf{-0.82999}$ ***Ans.***

Let $a = -3$ and $b = 0$ (for negative root)
$f(a).f(b) < 0$ **O.K.**
Let $x_L = a$ *and* $x_R = b$

$$x_c = \frac{x_L\, f(x_R) - x_R\, f(x_L)}{f(x_R) - f(x_L)}$$

n	x_L	x_R	$f(x_L)$	$f(x_R)$	x_c	$\lvert f(x_c)\rvert$
1	−3.000000	0.000000	−17.000000	1.000000	−0.166667	1.495370
2	−3.000000	−0.166667	−17.000000	1.495370	−0.395745	2.125255
3	−3.000000	−0.395745	−17.000000	2.125255	−0.685137	2.733799
4	−3.000000	−0.685137	−17.000000	2.733799	−1.005824	2.999898
5	−3.000000	−1.005824	−17.000000	2.999898	−1.304942	2.692675
6	−3.000000	−1.304942	−17.000000	2.692675	−1.536715	1.981202
⋮	⋮	⋮	⋮	⋮	⋮	⋮
26	−3.000000	−1.879384	−17.000000	0.000006	−1.879385	0.000003

The $-ve$ root is $\mathbf{-1.879385}$ because $|f(x_c^{26})| = 0.000003 < \alpha = 0.000005$
The minimum point is $x = \mathbf{-1.879385}$ and $y = f(-1.879385) = \mathbf{-5.058606}$ ***Ans.***

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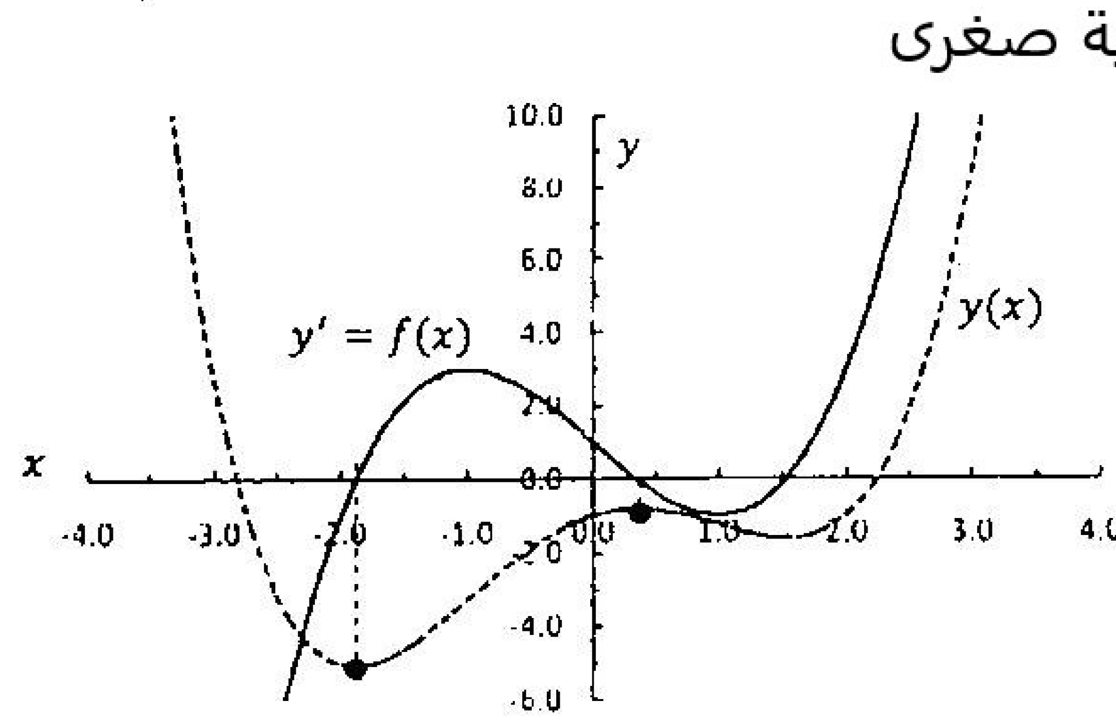

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5. Fixed-Point Method ($x = g(x)$ Method)

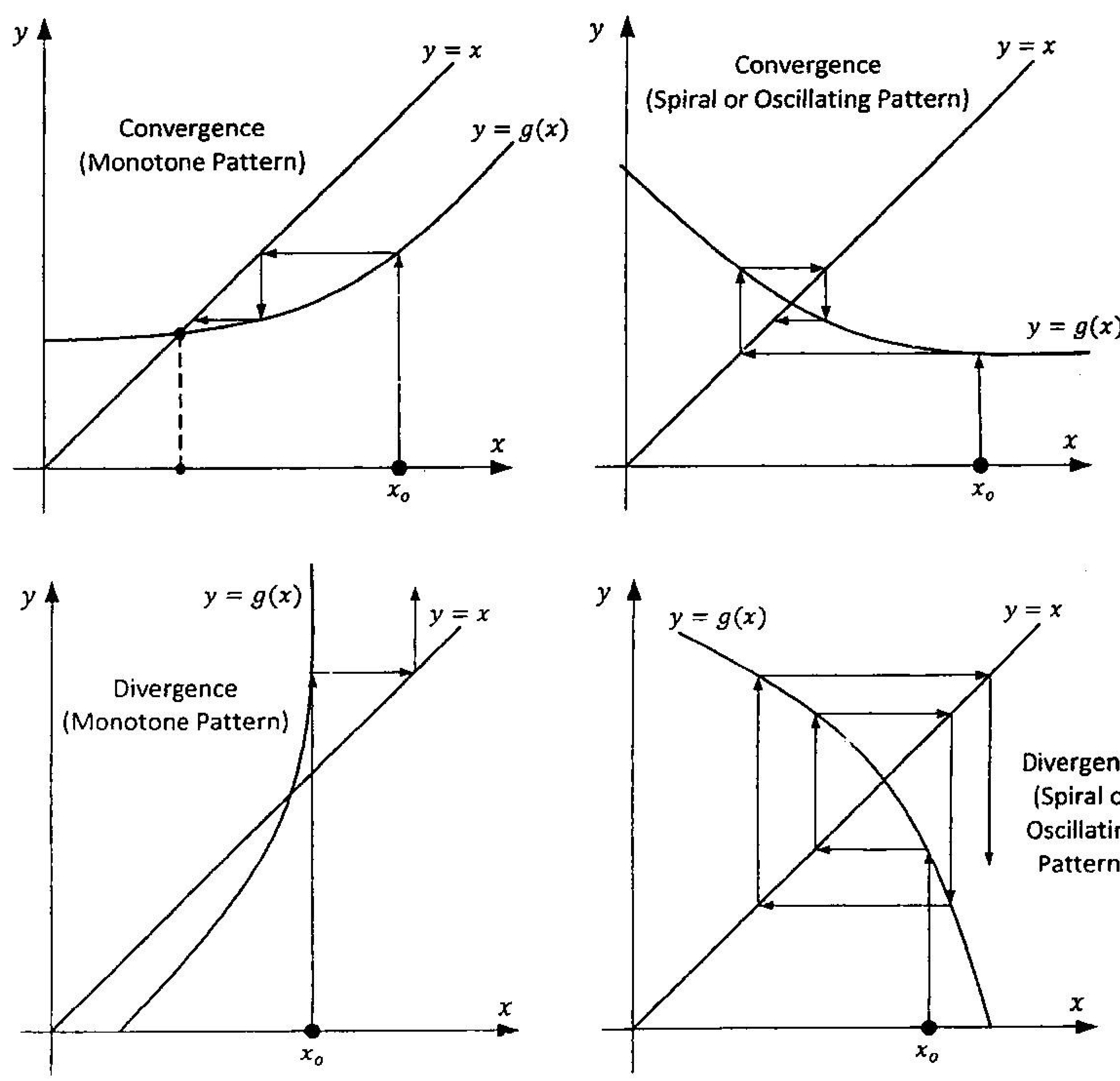
One can transform the equation $f(x) = 0$ algebraically into the form $x = g(x)$

Also, one can choose an arbitrary x_o and compute a sequence $x_o, x_1, x_2, \cdots, x_n$ from the relation

$$x_{n+1} = g(x_n)$$

This method is *problem dependent*. That is, $g(x)$ depends on the function $f(x)$. The solution path is shown in the figure below. This figure shows a graphical depiction of convergence or divergence for various $g(x)$ and various starting points x_o.

Note: An iteration process $x_n = g(x_n)$ is said to be convergent for an x_o if the corresponding sequence $x_o, x_1, x_2, \cdots, x_n$ convergent.



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Algorithm: $x = g(x)$ ***Method***

1. Given a function $f(x)$ real and continuous.
2. Transform $f(x)$ to $x = g(x)$ algebraically
3. Given a starting value x_o(initial guess).
4. Repeat the following steps until termination:
   a. If $f(x_n) = 0$, then the root is $x_n$and terminate the computation. Else,
   b. Compute

$$x_{n+1} = g(x_n),$$

   c. Test for termination (Termination Criteria):
      i. If $|x_m^{n+1} - x_m^n| \le \epsilon$ ($\epsilon > 0$, specified tolerance)
      ii. If $|f(x_m)| \le \alpha$ ($\alpha > 0$, specified tolerance)
      iii. After N steps (N, fixed)

Example 3.14

Determine approximate values of the roots of the equation $f(x) = x^2 - 3x + 1 = 0$, by using $x = g(x)$ Method. (*Correct to* 3*D*).

Solution

1. Let $3x = x^2 + 1 \Rightarrow x = \frac{1}{3}(x^2 + 1)$

$$x_{n+1} = \frac{1}{3}(x_n^2 + 1) = g_1(x_n)$$

Assume $x_o = 1.000$

n	0	1	2	3	4	5	6
x_{n+1}	0.667	0.481	0.411	0.390	0.384	0.382	0.382

The 1st positive root is 0.382 because $|x_6 - x_5| = 0$.

Assume $x_o = 2.000$

n	0	1	2	3	4	5	...	8	9
x_{n+1}	1.667	1.259	0.862	0.581	0.446	0.400	...	0.382	0.382

The same positive root 0.382 with 9 iterations.

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Assume $x_o = 3.000$

n	0	1	2	3	4	5	...	9
x_{n+1}	3.333	4.037	5.766	11.415	43.769	638.898	...	5.3714×10^{37}

There is no root. It seems to **diverge**.

2. The equation may also be written as

$x^2 - 3x + 1 = 0 \quad \div x$

$x - 3x + \frac{1}{x} = 0$

$$x_{n+1} = 3 - \frac{1}{x_n} = g_2(x_n)$$

Assume $x_o = 1.000$

n	0	1	2	3	4	5
x_{n+1}	2.000	2.500	2.600	2.615	2.618	2.618

The 2nd positive root is **2.618** because $|x_5 - x_4| = 0$.

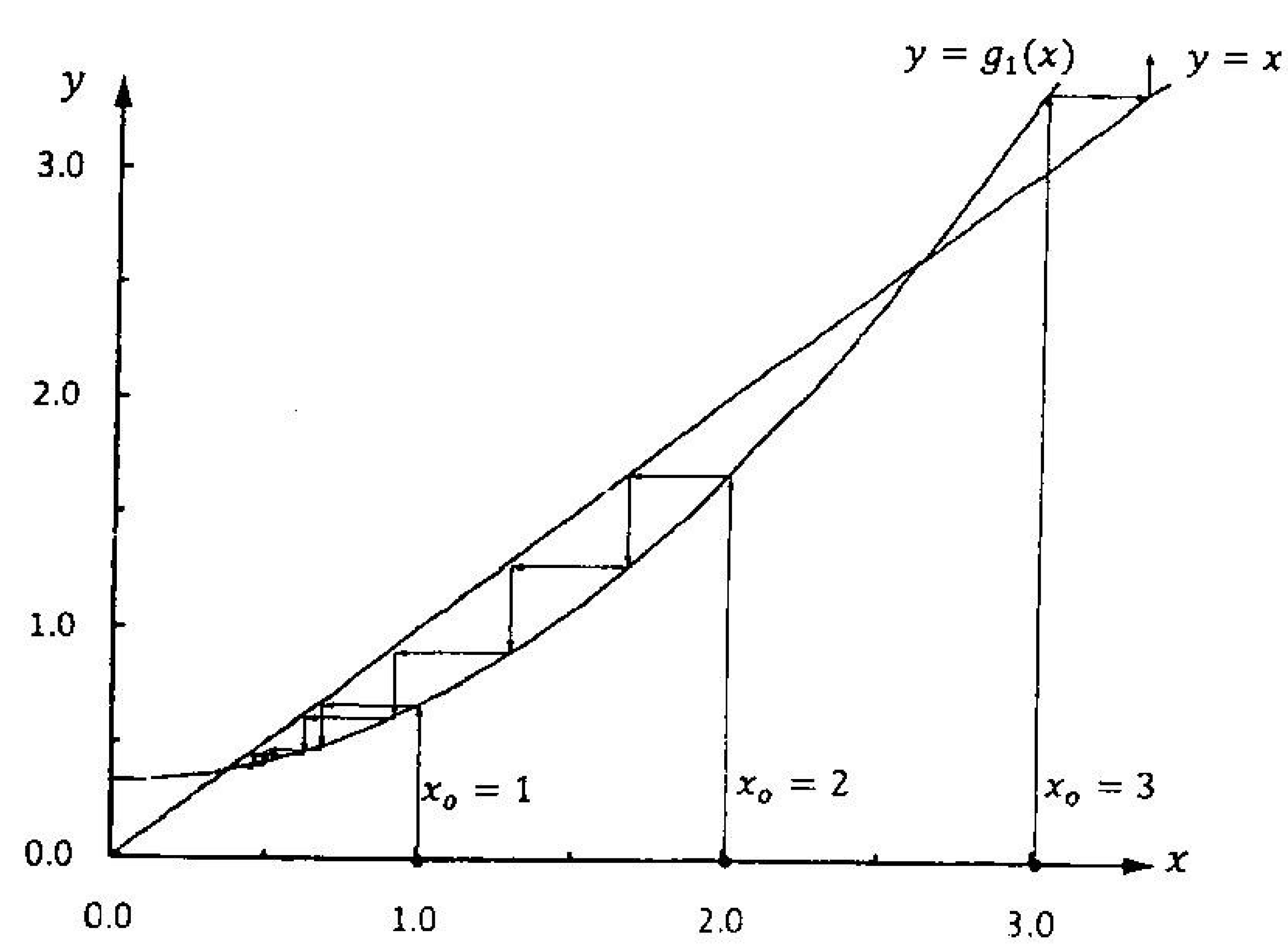
Assume $x_o = 2.000$

n	0	1	2	3	4
x_{n+1}	2.500	2.600	2.615	2.618	2.618

Assume $x_o = 3.000$

n	0	1	2	3	4
x_{n+1}	2.667	2.625	2.619	2.618	2.618

The solution is also the 2nd positive root **2.618** with $|x_4 - x_3| = 0$.



Graphical Depiction of $x_{n+1} = g_1(x_n)$

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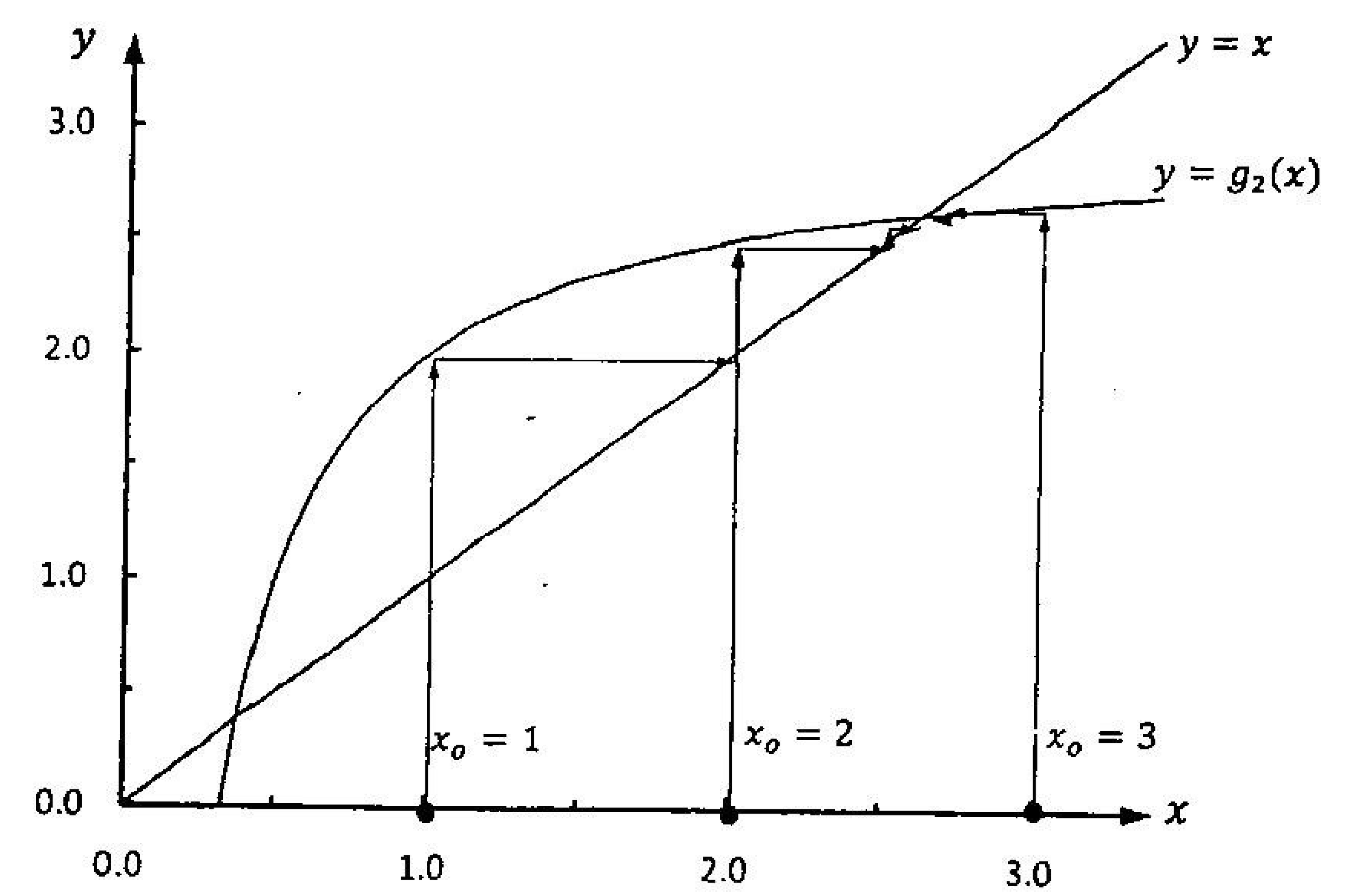


Graphical Depiction of $x_{n+1} = g_2(x_n)$

Example 3.15

Find the positive and negative roots of the equation $f(x) = x^3 - 3x^2 - 4x + 4$ using $x = g(x)$ Method. (*Correct to 4D*).

Solution

$f(x) = x^3 - 3x^2 - 4x + 4 = 0$ or $3x^2 = x^3 - 4x + 4$

then

$$g_{1,2}(x) = \pm\sqrt{\frac{x^3 - 4x + 4}{3}}$$

For positive root

$$g_1(x) = +\sqrt{\frac{x^3 - 4x + 4}{3}}$$

For negative root

$$g_2(x) = -\sqrt{\frac{x^3 - 4x + 4}{3}}$$

Chapter 2

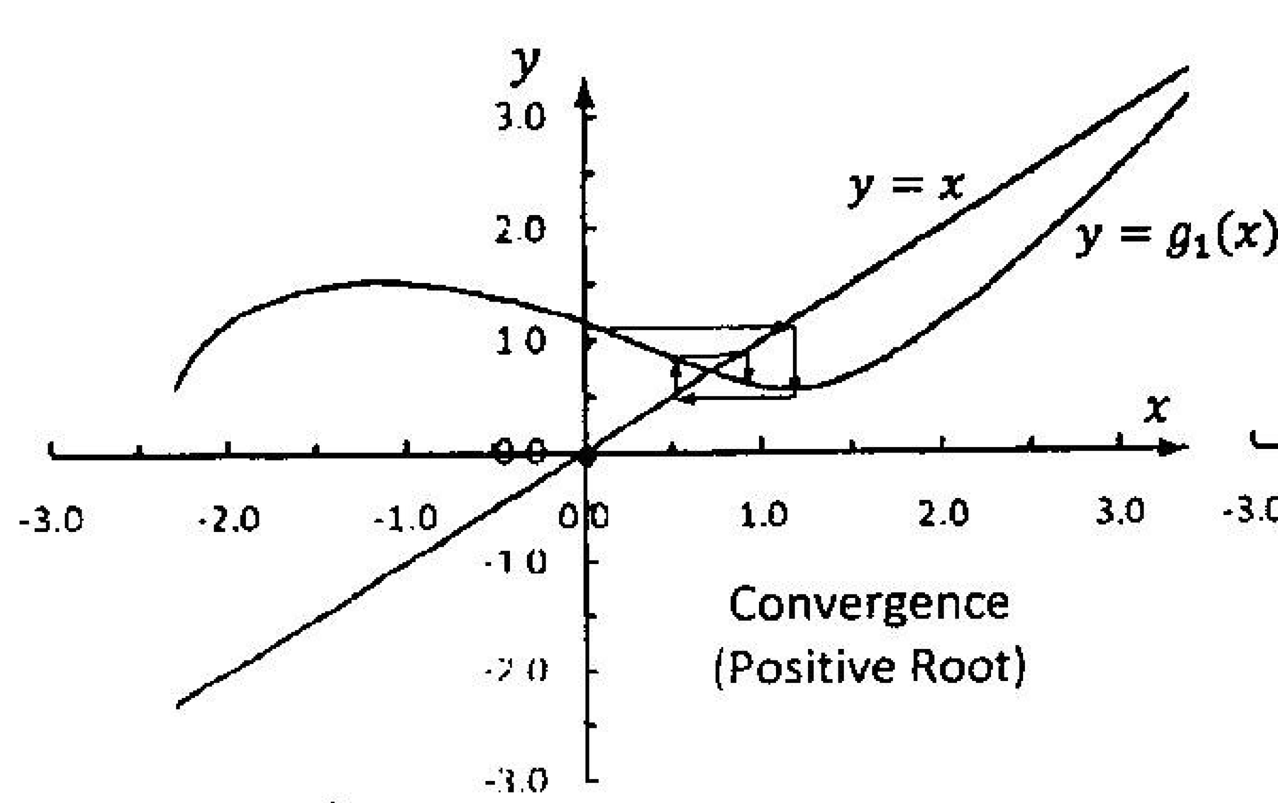
SOLUTION OF NONLINEAR EQUATIONS

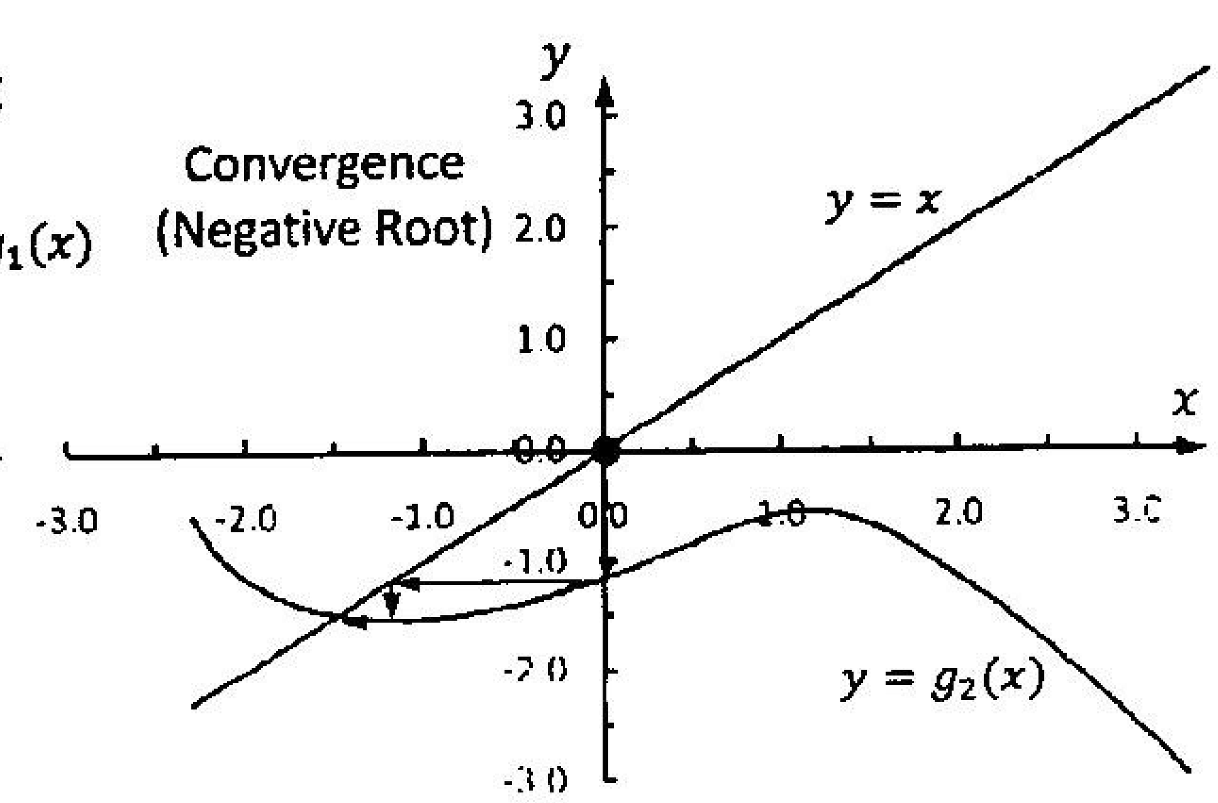
Let $xo = 0.0000$
For both $+ve$ and $-ve$ roots

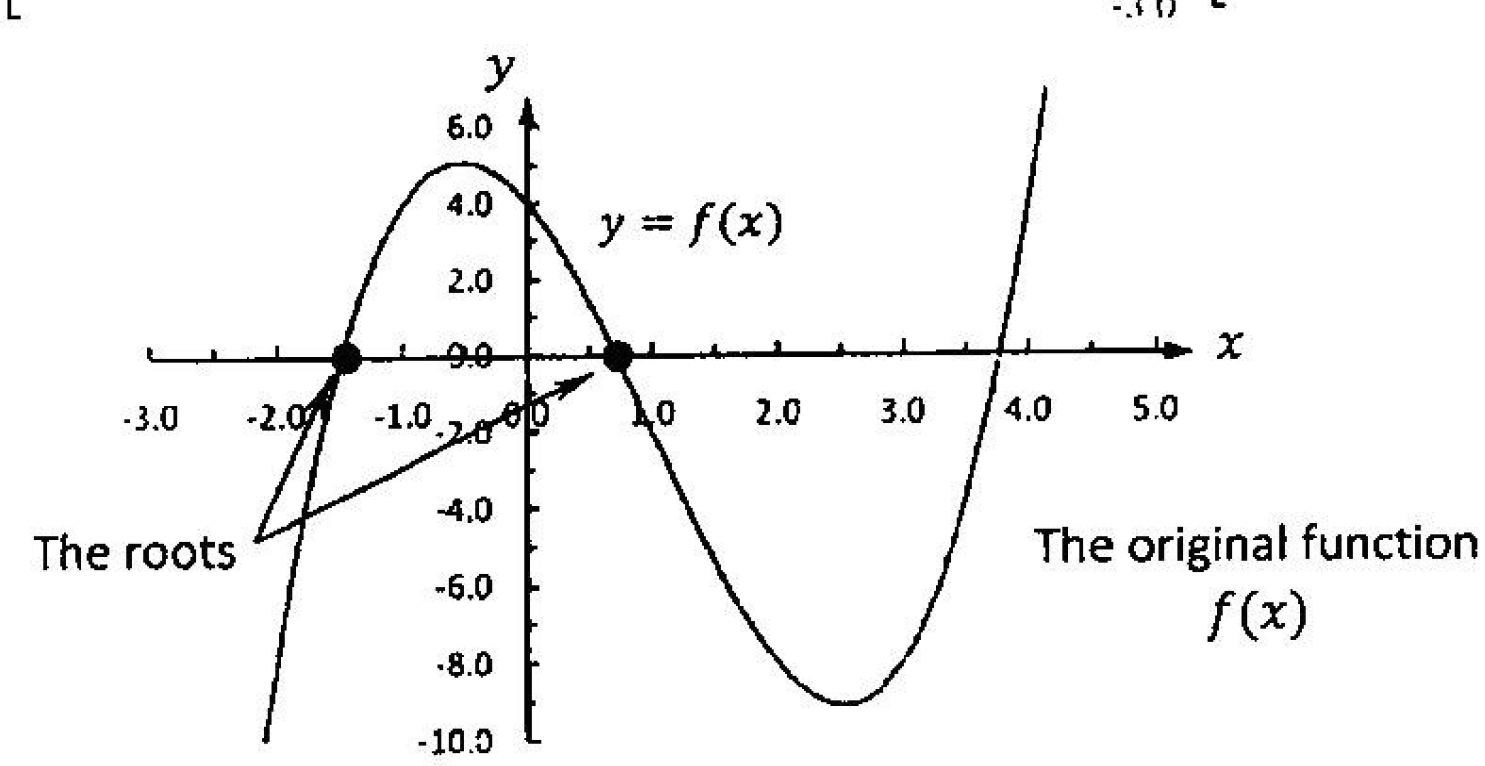
For $+ve\ root$, $g_1(x)$		
n	x_{n+1}	$f(x_{n+1})$
0	1.1547	−3.0792
1	0.554	1.0332
2	0.807	−0.6565
3	0.6576	0.3564
4	0.7425	−0.2145
5	0.6927	0.1223
6	0.7215	−0.0721
7	0.7047	0.0417
⋮	⋮	⋮
15	0.7107	0.0006
16	0.7109	−0.0003
17	0.7108	0.0002
18	0.7108	−0.0001

For $-ve\ root$, $g_2(x)$		
n	x_{n+1}	$f(x_{n+1})$
0	−1.1547	3.0792
1	−1.5361	−0.5595
2	−1.4742	0.1733
3	−1.4937	−0.0507
4	−1.488	0.0151
5	−1.4897	−0.0045
6	−1.4892	0.0013
7	−1.4893	−0.0004
8	−1.4893	0.0001

The positive and negative roots are **0.7108** and **−1.4893** because $|x_{n+1} - x_n| = 0.0000$ for $n = 17$ and 7 ,respectively. ***Ans.***







Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

SUMMARY OF SOLUTION OF NONLINEAR EQUATIONS

Method	Formula	Description
1. Graphical Method	-	Bracketing Method within an interval $[a, b]$ A simple method for obtaining an estimate of the root of the equation $f(x) = 0$ is to make a plot of the function and observe where it crosses the x axis. This point, which represents the x value for which $f(x) = 0$, provides a rough approximation of the root.
2. Bisection Method	$x_m = \frac{x_L + x_R}{2}$	Closed Method within an interval $[a, b]$ This simple (but slowly convergent method) is based on the intermediate value theorem for continuous functions.
3. Newton's Method	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $f'(x_n) \neq 0$	Open Method within an initial guess x_o This method is commonly used because of its simplicity and great speed. It depends on the function and its derivative.
4. False-Position Method	$x_c = \frac{x_L\ f(x_R) - x_R\ f(x_L)}{f(x_R) - f(x_L)}$	Closed Method within an interval $[a, b]$ In this method one can approximate the curve of $f(x)$ by a chord between a and b (*i.e.*, linear interpolation).
5. $x = g(x)$ Method	$x_{n+1} = g(x_n)$	Open Method within an initial guess x_o In this method one can transform the equation $f(x) = 0$ algebraically into the form $x = g(x)$.

Termination Criteria

i. If $|x_m^{n+1} - x_m^n| \leq \epsilon$ ($\epsilon > 0$, specified tolerance)

ii. If $|f(x_m)| \leq \alpha$ ($\alpha > 0$, specified tolerance)

iii. After N steps (N, fixed)

Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

HOME WORKS: SOLUTION OF NONLINEAR EQUATIONS

The Graphical Method

H.W 3.1

Determine the root(s) of $f(x) = e^x - 3x^2$ graphically by using MATLAB with $2D$ accuracy.

Answer: $x_{1,2,3} = -0.46,\ 0.91$ and 3.73

H.W 3.2

Determine the root(s) of $f(x) = -2 + 6.2x - 0.4x^2 + 0.7x^3$ graphically by using MATLAB with $2D$ accuracy.

Answer: $x = 0.33$

H.W 3.3

Determine the root(s) of $f(x) = \frac{1-0.61x}{x}$ graphically by using MATLAB with $2D$ accuracy.

Answer: $x = 1.64$

H.W 3.4

Determine the root(s) of $-x^2 \sin x = 4.1, [-15, 15]$ graphically by using MATLAB with $2D$ accuracy.

Answer: $x_{1-7} = -12.59, -9.38, -6.39, 3.49, 6.18, 9.47$ and 12.54

H.W 3.5

Determine the root(s) $e^{-x}\cos x - 2x = 0\ [-5, 1]$ graphically by using MATLAB with $2D$ accuracy.

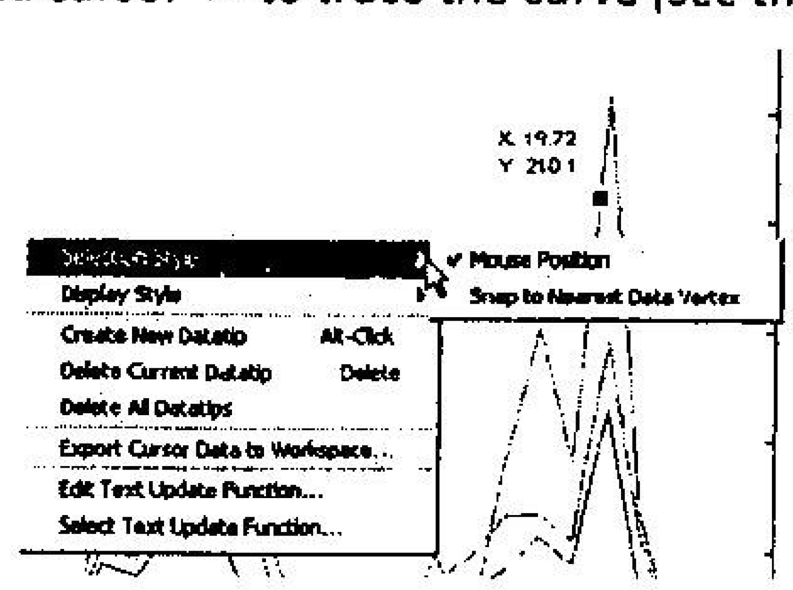
Answer: $x_{1-3} = -4.62, -2.11$ and 0.34

H.W 3.6

Determine the root(s) $\sin^{-1}(x + 2x^2) = \cos x$ graphically by using MATLAB with $2D$ accuracy.

Answer: $x_{1,2} = -0.86$ and 0.43

Hint: Plot with $\Delta x = 0.01$ *and use data cursor to trace the curve (see the figure below)*



Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

The Bisection Method

H.W 3.7

The velocity of a falling parachutist is given by

$$v = \frac{gm}{c}\left[1 - e^{-(c/m)t}\right]$$

where $g = 9.81$ for a parachutist with a drag coefficient c equals $13.5\ kg/s$, compute the mass m so that the velocity is $v = 36\ m/s$ at $t = 6\ sec$. Use the Bisection Method to determine m with two decimals accuracy ($\alpha_{max} = 0.06$). (*Hint: Assume reasonable mass interval and reasonable accuracy*).

Answer: $m = 75\ kg$

H.W 3.8

Find the maximum point of the function

$$f(x) = \sin^6 x \,.\, e^{20x}.\tan(1 - x)$$

on the interval $[0,1]$ by using the Bisection Method. (*Note: Correct to 3D*)

Answer: $x = 0.959, y = 2.637 \times 10^6$

H.W 3.9

Find the intersection points between the functions

$$y_1 = \sin x \quad \text{and} \quad y_2 = \cos x\ e^{-x}$$

within the interval $[-8, -4]$ by using the Bisection Method. (*Note: Correct to 6D and N_{max}=10*)

Answer: $(x_{1,2}, y_{1,2}) = (-4.720703, 1.)$ and $(-7.853516, -1.)$

Newton's Method

H.W 3.10

Find the roots of the function

$$x^2 = 2^x \cos(0.5x)$$

within the interval from $x = -4$ to $x = 10$ by using Newton's Method. (*Correct to 6D*).

Answer: $x_{1,2,3} = -0.745365, 1.425562$ and 9.656393

H.W 3.11

Find all max/min points of the function

$$f(x) = \frac{\sin x}{x} - \cos x$$

on the interval $[0,10]$ by using the Newton's Method. (*Note: Correct to 6D*)

Answer: $(x_{1,2,3}, y_{1,2,3}) = (2.743707, 1.063104), (6.116764, -1.013266)$ and $(9.316616, 1.005743)$

Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

H.W 3.12

Use Newton's Method to find t if

$$\sin t + 2t = 3\cos 2t$$

assume $t_o = 5.0$ (starting point) and correct to 5D.

Answer: $t = 0.51773$

The False-Position Method

H.W 3.13

For the following relationship between z and y:

$$z = \frac{1 + y + y^2 - y^3}{(1-y)^3}$$

What is the value of y if $z = 0.892$? Use the False-Position Method to solve for y.

(Note: *the value of y is between* 0 *and* 3, *correct to 4D and use* $\epsilon_{max} = 0.08$).

Answer: $y = 1.9240$

H.W 3.14

Find the positive root of the equation

H.W 3.17

Find the solution of the function

$$2^x = 3x^2 - 2$$

by using $x = g(x)$ Method with four decimals accuracy. (Use $x_o = 0$).

Answer: $x_{1,2} = -0.9182$ and 1.1960

H.V

H.W 3.18

Det

Find the roots of the function

$$f(w) = w^3 + e^{-w} - 3w$$

witl

by using $x = g(x)$ Method with four decimal places.

Answer: $w_1 = 0.2740$ and $w_2 = 1.7008$ 53

$x = g(x)$ Method

H.W 3.16

Find the roots of the equation

$$f(x) = e^x - 3x^2$$

by using $x = g(x)$ Method with six decimals accuracy. (Use $x_o = 1$).

Answer: $x_{1,2,3} = -0.458962,\ 0.910008$ and 3.733079

H.W 3.17

Find the solution of the function

$$2^x = 3x^2 - 2$$

by using $x = g(x)$ Method with four decimals accuracy. (Use $x_o = 0$).

Answer: $x_{1,2} = -0.9182$ and 1.1960

H.W 3.18

Find the roots of the function

$$f(w) = w^3 + e^{-w} - 3w$$

by using $x = g(x)$ Method with four decimal places.

Answer: $w_1 = 0.2740$ and $w_2 = 1.7008$

Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

Case Study 3.8 (Theory of Structures)

Figure 3.8 shows a uniformly beam subjected to a linearly increasing distributed load. The equation for the resulting elastic curve is

$$y = \frac{w}{120EIL}(-x^5 + 2L^2x^3 - L^4x)$$

Determine the point of maximum deflection (that is, the value of x where $dy/dx = 0$). Then determine the value of maximum deflection. Use $L = 180\ in, E = 29 \times 10^6\ lb/in^2, I = 723\ in^4$ and $w = 12\ kips/ft$. Express your results in inches.

Note: Use an appropriate method and significant accuracy.

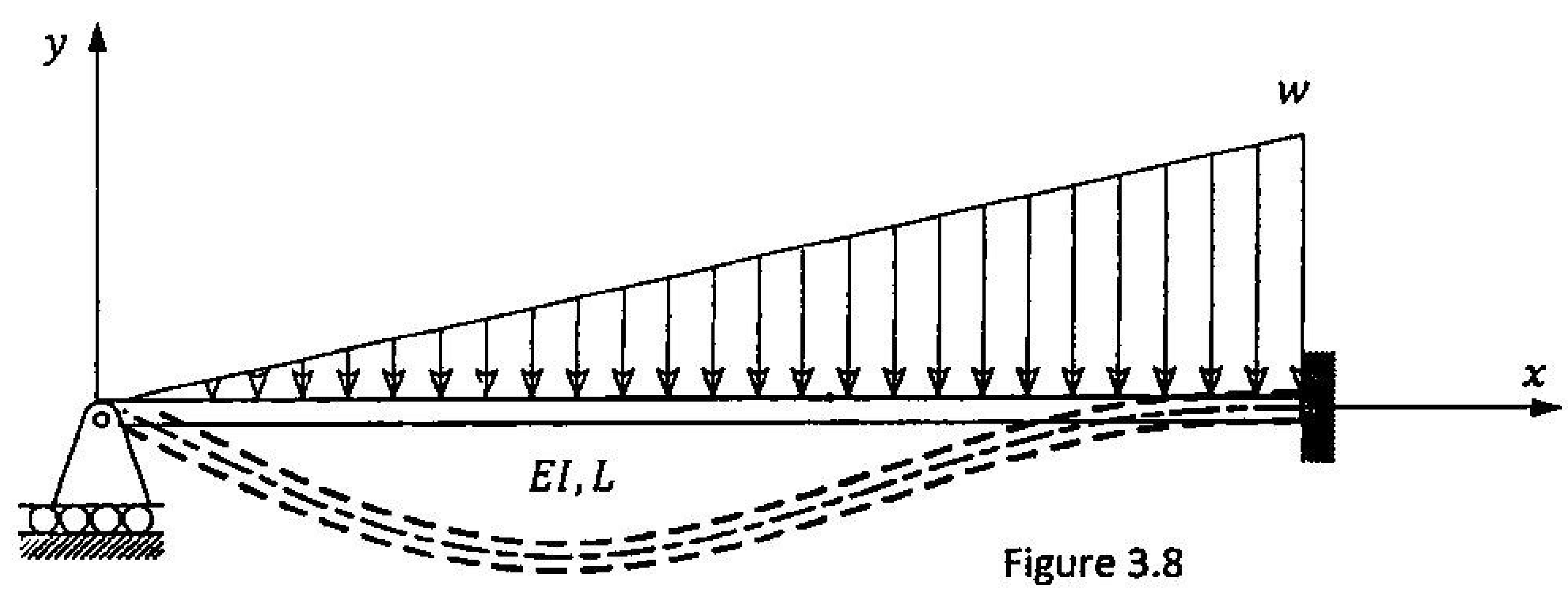


Figure 3.8

Case Study 3.9 (Ocean Engineering)

In ocean engineering, the equation for a reflected standing wave in a harbor is given by

$$h = h_o\left[\sin\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi tv}{\lambda}\right) + e^{-x}\right]$$

Solve for x if $h = 0.5h_o, \lambda = 20, t = 10$ and $v = 50$.

Note: Use an appropriate method and significant accuracy.

Case Study 3.10 (Strength of Materials)

The secant formula defines the force per unit area , P/A. That causes a maximum stress σ_m in a column of given slenderness ratio L_e/r:

$$\frac{P}{A} = \frac{\sigma_m}{1 + (ec/r^2)\sec\left[1/2(\sqrt{P/EA})(L_e/r)\right]}$$

If $E = 29 \times 10^3\ ksi, ec/r^2 = 0.2$ and $\sigma_m = 36\ ksi$. Compute P/A for $L_e/r = 100$ (*Hint: Recall that* $\sec x = 1/\cos x$).

Note: Use an appropriate method and significant accuracy.

Higher Education & Scientific Research Ministry

Al-Mustafa University College

Civil Engineering Department

Numerical Analysis

Third Stage

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Chapter 3

SYSTEMS OF LINEAR EQUATIONS

A system of n linear equations (or a set of n simultaneous linear equations) in n unknowns $x_1, x_2, \dots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Where the coefficients a_{ij} and b_i are given numbers. If all the b_i are zeros, the system is homogenous, otherwise, the system is nonhomogenous.

The above system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix} \Leftrightarrow \mathbf{Ax} = \mathbf{b}$$

where **A** the coefficients matrix, **x** is the unknowns vector and **b** is the right side vector.

The solution of the above system is

$$\mathbf{A^{-1}Ax} = \mathbf{A^{-1}b} \Rightarrow \mathbf{Ix} = \mathbf{A^{-1}b}$$

$$\mathbf{x} = \mathbf{A^{-1}b}$$

in which $\mathbf{A^{-1}}$ is the inverse of **A** and **I** is an identity matrix.

1. The Graphical Method

A graphical method is obtained for two equations by plotting them on Cartesian coordinates with one axis corresponding to x_1 and other to x_2. The equations obtained are straight lines because we are dealing with linear systems. The general matrix form of two linear equations is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$$

Both equations can be solve for x_2 as

$$x_2 = -\left(\frac{a_{11}}{a12}\right) x_1 + \frac{c_1}{a_{12}}$$

$$x_2 = -\left(\frac{a_{21}}{a22}\right) x_1 + \frac{c_2}{a_{22}}$$

Chapter 3

SYSTEMS OF LINEAR EQUATIONS

Thus, the equations are now in the form of straight lines; that is

$$x_2 = (slope)x_1 + intercept$$

These lines can be graphed on Cartesian coordinates with x_2 as the ordinate and x_1 as the abscissa. The values of x_1 and x_2 at the intersection of the lines represents the solution.

For three simultaneous equations, each equation would be represented by a plane in three-dimensional coordinate system. The point where the planes intersect would represent the solution.

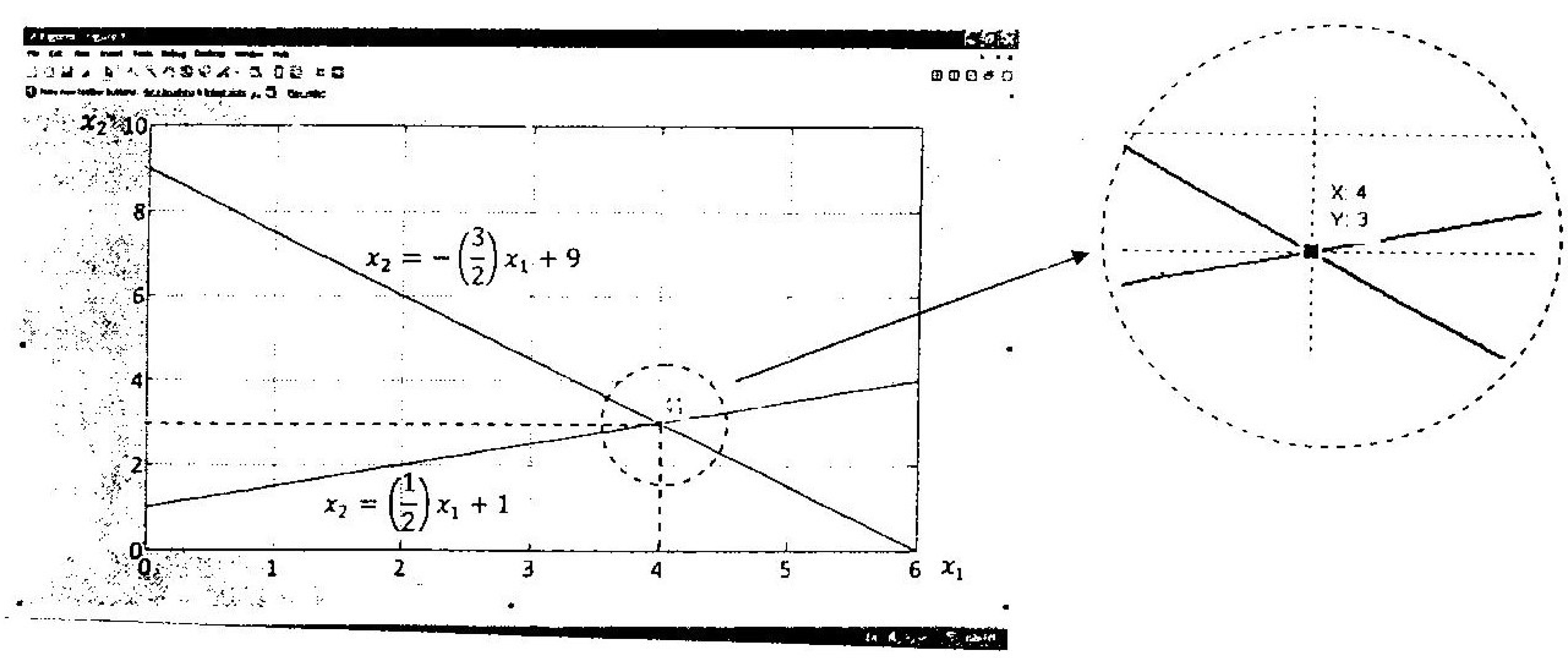
Example 4.1

Use the *graphical method* to solve the system

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

Solution

$$x_2 = -\left(\frac{3}{2}\right)x_1 + 9$$
$$x_2 = \left(\frac{1}{2}\right)x_1 + 1$$

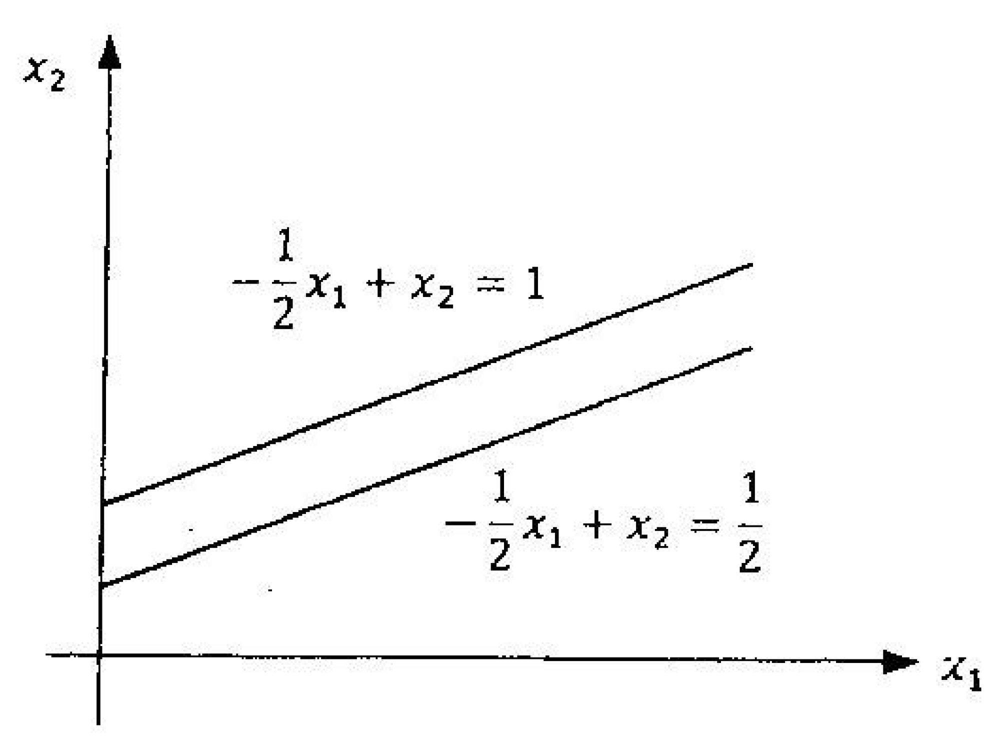


we can solve by Excel

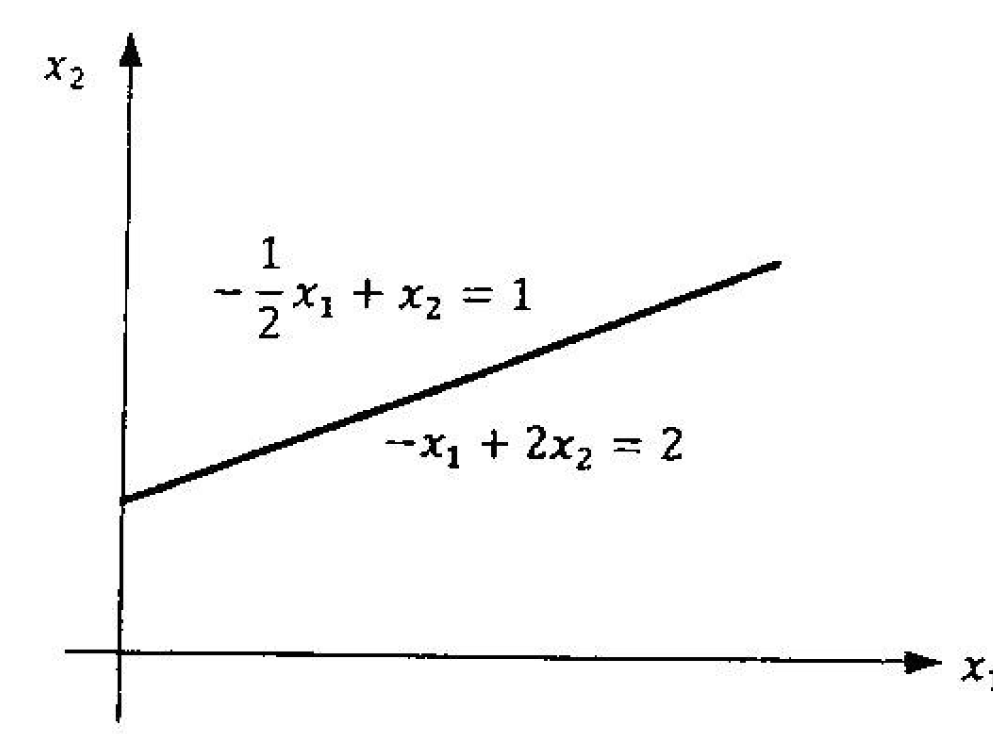
Chapter 3

SYSTEMS OF LINEAR EQUATIONS

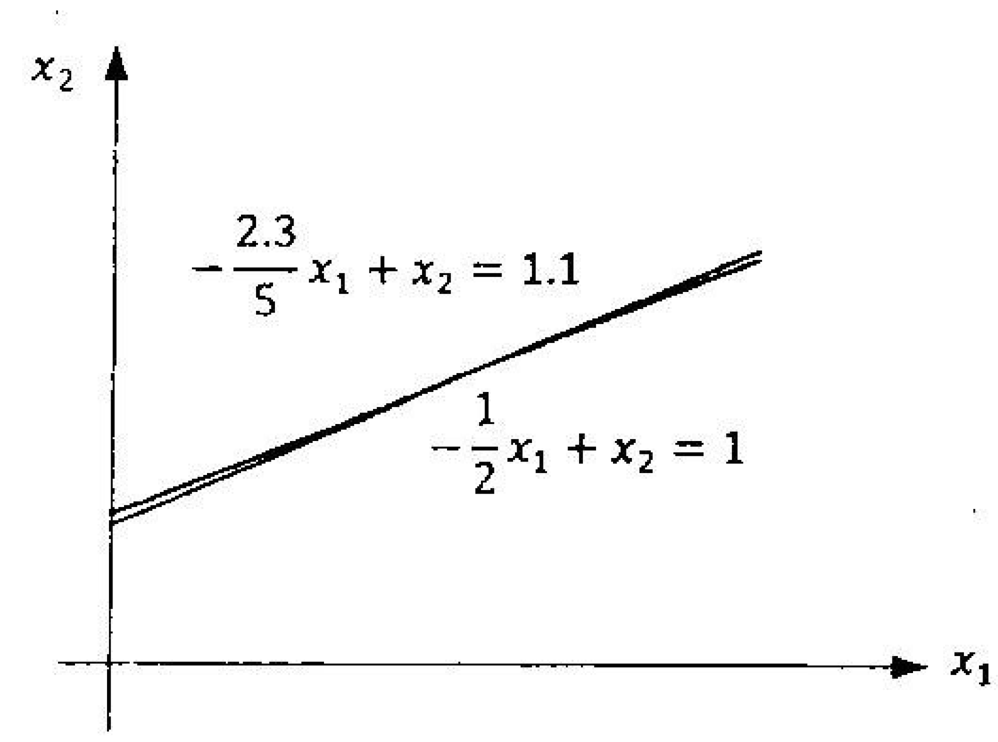
It is worthwhile to mention that some systems are said to be ***ill-conditioned***, that is, it is difficult to identify the exact point at which the lines intersect. Ill-conditioned systems will also pose problems when they are encountered during the numerical solution of linear equations. This is due to the fact that they will be extremely sensitive to round-off error. The below figures show graphical depiction of singular and ill-conditioned systems.



No Solution



Infinite Solution



Ill-Conditioned System where the slopes are so close that the point of intersection is difficult to detect visually

Chapter 3

SYSTEMS OF LINEAR EQUATIONS

2. Gaussian Elimination Method (Matrix Inversion Method or Direct Method)

The following algebraic approach to element unknowns by combining equations will be useful to illustrate Gaussian Elimination Method.

For the following set of four linear equations,

$$2w + x + 2y + z = 6 \qquad \cdots(1)$$

$$6w - 6x + 6y + 12z = 36 \qquad \cdots(2)$$

$$4w + 3x + 3y - 3z = -1 \qquad \cdots(3)$$

$$2w + 2x - y + z = 10 \qquad \cdots(4)$$

the basic strategy is reduce the set of equations from 4×4 to 1×1. This will be done by eliminating the unknowns w, x and y in the following three steps.

First Step (Eliminate the unknown w)

$Eq.(5) = Eq.(2) - q_{11} \times Eq.(1)$ $\qquad q_{11} = \frac{6}{2} = \frac{a_{21}}{a_{11}}$

$Eq.(6) = Eq.(3) - q_{12} \times Eq.(1)$ $\qquad q_{12} = \frac{4}{2} = \frac{a_{31}}{a_{11}}$

$Eq.(7) = Eq.(4) - q_{13} \times Eq.(1)$ $\qquad q_{13} = \frac{2}{2} = \frac{a_{41}}{a_{11}}$

the equations will be reduced from 4×4 to 3×3 as follows:

$$-9x + 9z = 18 \qquad \cdots(5)$$

$$x - y - 5z = -13 \qquad \cdots(6)$$

$$x - 3y = 4 \qquad \cdots(7)$$

Second Step (Eliminate the unknown x)

$Eq.(8) = Eq.(6) - q_{21} \times Eq.(5)$ $\qquad q_{21} = -\frac{1}{9} = \frac{a_{32}}{a_{22}}$

$Eq.(9) = Eq.(7) - q_{22} \times Eq.(5)$ $\qquad q_{22} = -\frac{1}{9} = \frac{a_{42}}{a_{22}}$

the equations will be reduced from 3×3 to 2×2 as follows:

$$-y - 4z = -11 \qquad \cdots(8)$$

$$-3y + z = 6 \qquad \cdots(9)$$

Chapter 3

SYSTEMS OF LINEAR EQUATIONS

<u>Third Step (Eliminate the unknown y)</u>

$Eq.(10) = Eq.(9) - q_{31} \times Eq.(8)$ $\qquad q_{31} = \frac{3}{1} = \frac{a_{32}}{a_{22}}$

the equations will be reduced from 2×2 to 1×1 as follows:

$$13z = 39 \qquad \cdots (10)$$

<u>Final Step (Back-Substitution)</u>

From Eq. (10): $13z = 39 \qquad \Rightarrow \quad z = 3$

From Eq. (8) : $-y - 4(3) = -11 \qquad \Rightarrow \quad y = -1$

From Eq. (5) : $-9x + 9(3) = 18 \qquad \Rightarrow \quad x = 1$

From Eq. (1) : $2w + (1) + 2(-1) + (3) = 6 \quad \Rightarrow \quad w = 2$

To visualize the above mathematical manipulation, rewrite Eqs. (1) to (4) into the following matrix form

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix} \begin{Bmatrix} w \\ x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{Bmatrix}$$

also, one can rewrite Eqs. (1), (5), (8) and (10) into the following matrix form

Lower Triangle

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & -9 & 0 & 9 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 13 \end{bmatrix} \begin{Bmatrix} w \\ x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 6 \\ 18 \\ -11 \\ 39 \end{Bmatrix}$$

Major Diagonal

Therefore, the eliminating strategy in matrix-sense can be done by zeroing the coefficients in lower major triangle (the triangle under the major diagonal).

Chapter 3

SYSTEMS OF LINEAR EQUATIONS

Algorithm: Gaussian Elimination Method

1. Given a $\mathbf{Ax} = \mathbf{b}$, $\mathbf{A} = [a_{ij}]$ an $n \times n$ matrix and $\mathbf{b} = \{b_i\}$ an n vector.
2. For $k = 1, \cdots, n-1$, do:

   If $a_{jk} = 0$ for all $j \geq k$ then Stop. "No Unique Solution"

3. Else exchange the contents of rows j and k of $\mathbf{A}$ with j the smallest $j \geq k$ such that $|a_{jk}|$ is maximum in column k.
4. For $j = k+1, \cdots, n$, do:

   $$q_{jk} = \frac{a_{jk}}{a_{kk}}, b_j = b_j - q_{ik} \times b_k$$

   For $p = 1, \cdots, n$, do:

   $$a_{jp} = a_{jp} - q_{jk}\, a_{kp}$$

   End

   End

   End
5. If $a_{nn} = 0$ then Stop. "No Unique Solution"

   Else, continue with Back-Substitution
6. $$x_n = \frac{a_{n,n+1}}{a_{nn}}$$
7. For $i = n-1, \cdots, 1$, do:

   $$x_i = \frac{1}{a_{ii}}\left(a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j\right)$$

   End

   Output $\mathbf{x}$

Example 4.2

Solve the system

$$3x_1 - 6x_2 + 7x_3 = 3$$
$$9x_1 \qquad - 5x_3 = 3$$
$$5x_1 - 8x_2 + 6x_3 = -4$$

by using Gaussian Elimination Method. (Correct to $3D$ and check for accuracy).

Solution

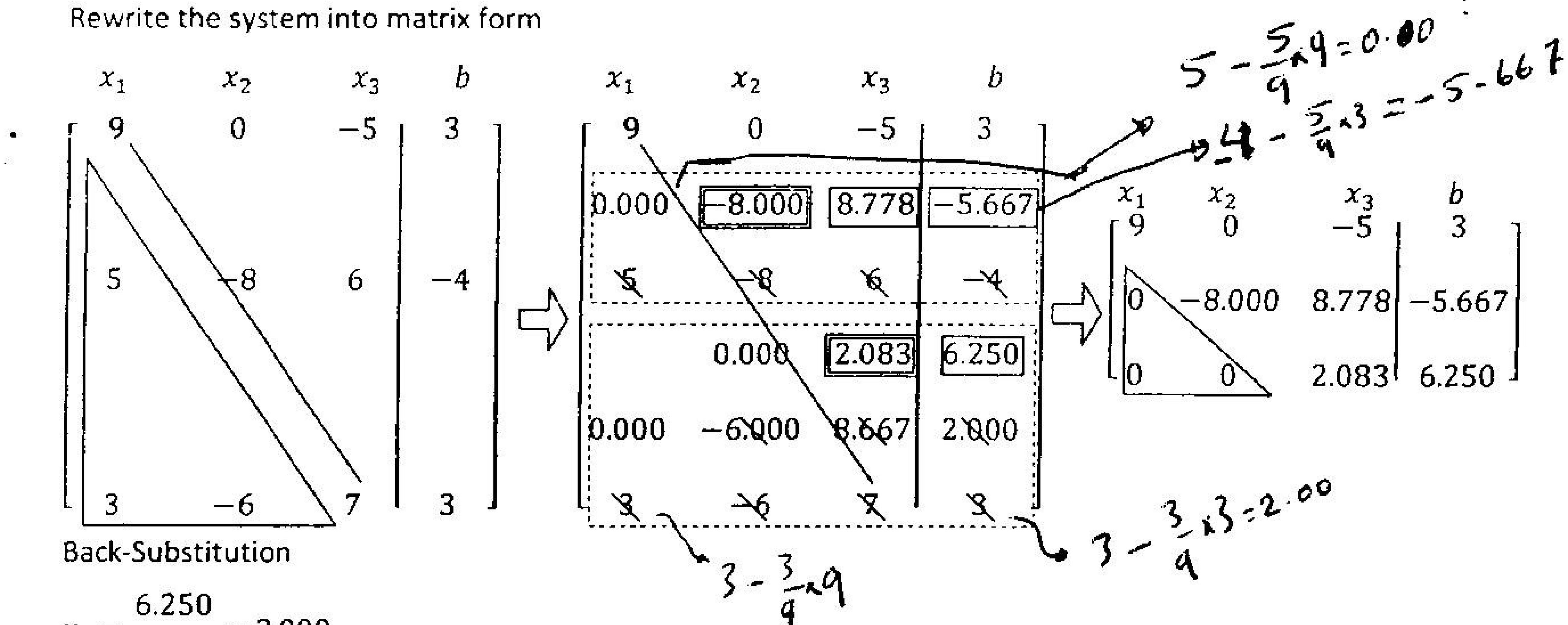
Rearrange the system

Chapter 3

SYSTEMS OF LINEAR EQUATIONS

$$9x_1 \qquad - 5x_3 = 3 \qquad \cdots(1)$$
$$5x_1 - 8x_2 + 6x_3 = -4 \qquad \cdots(2)$$
$$3x_1 - 6x_2 + 7x_3 = 3 \qquad \cdots(3)$$

Rewrite the system into matrix form



Back-Substitution

$$x_3 = \frac{6.250}{2.083} = 3.000$$

$$x_2 = \frac{1}{-8}[-5.667 - (8.778 \times 3.000)] = 4.000$$

$$x_1 = \frac{1}{9}[3 - (0 \times 4.000 - 5 \times 3.000)] = 2.000$$

Then

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4 \\ 3 \end{Bmatrix}$$ **Ans.**

Check for accuracy

Sub. into Eq. (1): $9(2) - 5(3) = 3.000$ **O.K.**

Sub. into Eq. (2): $5(2) - 8(4) + 6(3) = -4.000$ **O.K.**

Sub. into Eq. (3): $3(2) - 6(4) + 7(3) = 3.000$ **O.K.**

Example 4.3

Solve the system

$$5.8y + 0.62z + 2.28w = 22.58$$
$$3.25x + 1.35y + 2.5z + 0.75w = 16.45$$
$$-3x + 2.25y - 3.25z + 8w = 23.75$$
$$4.22y + 6.65z + 2.5w = 38.39$$

by using Gaussian Elimination Method. (Correct to $3D$ and check for accuracy).

Chapter 3

SYSTEMS OF LINEAR EQUATIONS

Solution

Rearrange the system

$$3.25x + 1.35y + 2.5z + 0.75w = 16.45 \qquad \cdots (1)$$
$$5.8y + 0.62z + 2.28w = 22.58 \qquad \cdots (2)$$
$$4.22y + 6.65z + 2.5w = 38.39 \qquad \cdots (3)$$
$$-3x + 2.25y - 3.25z + 8w = 23.75 \qquad \cdots (4)$$

Rewrite the system into matrix form

$$\begin{array}{cccc|c} x & y & z & w & b \\ 3.25 & 1.35 & 2.5 & 0.75 & 16.45 \\ 0 & 5.8 & 0.62 & 2.28 & 22.58 \\ 0 & 4.22 & 6.65 & 2.5 & 38.39 \\ -3 & 2.25 & -3.25 & 8 & 23.75 \end{array}$$

$$\begin{array}{cccc|c} x & y & z & w & b \\ 3.25 & 1.35 & 2.5 & 0.75 & 16.45 \\ 0 & 5.8 & 0.62 & 2.28 & 22.58 \\ & 0.000 & 6.199 & 0.841 & 21.961 \\ 0 & 4.22 & 6.65 & 2.5 & 38.39 \\ & & 0.000 & 7.497 & 29.987 \\ & 0.000 & -1.316 & 7.318 & 25.325 \\ 0.000 & 3.496 & -0.942 & 8.692 & 38.935 \\ -3 & 2.25 & -3.25 & 8 & 23.75 \end{array}$$

$$\begin{array}{cccc|c} x & y & z & w & b \\ 3.25 & 1.35 & 2.5 & 0.75 & 16.45 \\ 0 & 5.8 & 0.62 & 2.28 & 22.58 \\ 0 & 0 & 6.199 & 0.841 & 21.961 \\ 0 & 0 & 0 & 7.497 & 29.987 \end{array}$$

Back-Substitution

$$w = \frac{29.987}{7.497} = 4.000$$

$$z = \frac{1}{6.199}[21.961 - (0.841 \times 4)] = 3.000$$

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$$y = \frac{1}{5.8}[22.58 - (0.62 \times 3 + 2.28 \times 4)] = 2.000$$

$$x = \frac{1}{3.25}[16.45 - (1.35 \times 2 + 2.5 \times 3 + 0.75 \times 4)] = 1.000$$

Then

$$\begin{Bmatrix} x \\ y \\ z \\ w \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{Bmatrix} \quad \underline{\text{Ans.}}$$

Check for accuracy

Sub. into Eq. (1): $3.25(1) + 1.35(2) + 2.5(3) + 0.75(4) = 16.450$ **O.K.**

Sub. into Eq. (2): $5.8(2) + 0.65(3) + 2.28(4) = 22.580$ **O.K.**

Sub. into Eq. (3): $4.22(2) + 6.65(3) + 2.5(4) = 38.390$ **O.K.**

Sub. into Eq. (4): $-3(1) + 2.25(2) - 3.25(3) + 8(4) = 23.750$ **O.K.**

Example 4.4

Solve the system

$$\begin{aligned} 2x_2 \qquad\qquad + x_4 &= 0 \\ 6x_1 + x_2 - 6x_3 - 5x_4 &= 6 \\ 2x_1 + 2x_2 + 3x_3 + 2x_4 &= -2 \\ 4x_1 - 3x_2 + \qquad\quad x_4 &= -7 \end{aligned}$$

by using Gaussian Elimination Method. (Correct to $3D$ and check for accuracy).

Solution

Rearrange the system

$$\begin{aligned} 6x_1 + x_2 - 6x_3 - 5x_4 &= 6 && \cdots(1) \\ 4x_1 - 3x_2 + \qquad\quad x_4 &= -7 && \cdots(2) \\ 2x_1 + 2x_2 + 3x_3 + 2x_4 &= -2 && \cdots(3) \\ 2x_2 \qquad\qquad + x_4 &= 0 && \cdots(4) \end{aligned}$$

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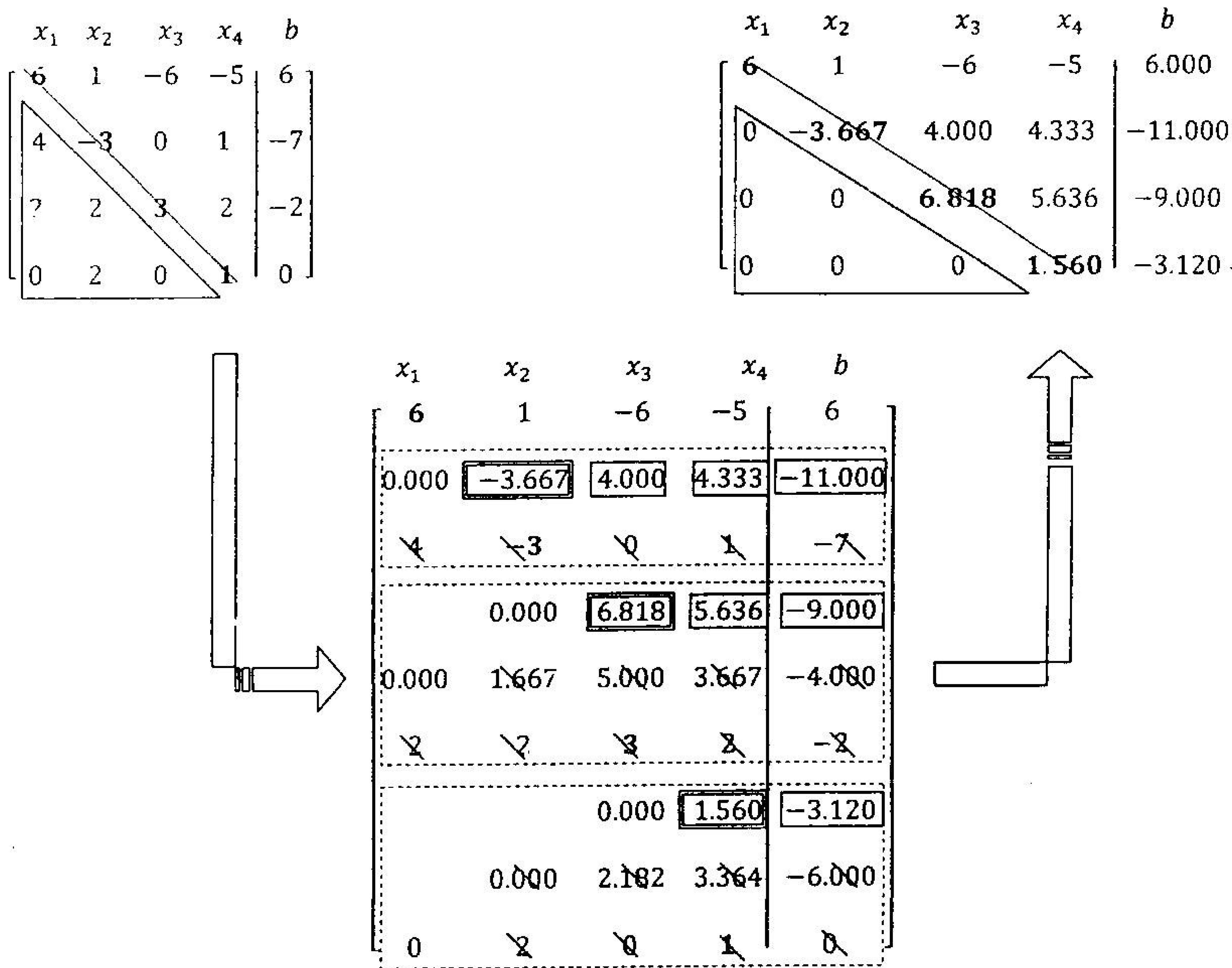
Rewrite the system into matrix form



Back-Substitution

$$x_4 = \frac{-3.120}{1.560} = -2.000$$

$$x_3 = \frac{1}{6.818}[-9 - (-2 \times 5.636)] = 0.333$$

$$x_2 = \frac{1}{-3.667}[-11 - (0.333 \times 4 + 4.333 \times -2] = 1.000$$

$$x_1 = \frac{1}{6}[6 - (1 \times -5 - 6 \times 0.333 - 5 \times -2)] = -0.500$$

Then

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} -0.500 \\ 1.000 \\ 0.333 \\ -2.000 \end{Bmatrix} \ \underline{\textbf{Ans.}}$$

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Check for accuracy

Sub. into Eq. (1): $6x_1 + x_2 - 6x_3 - 5x_4 = 6.002$

Sub. into Eq. (2): $4x_1 - 3x_2 + x_4 = -7.000$

Sub. into Eq. (3): $2x_1 + 2x_2 + 3x_3 + 2x_4 = -2.001$

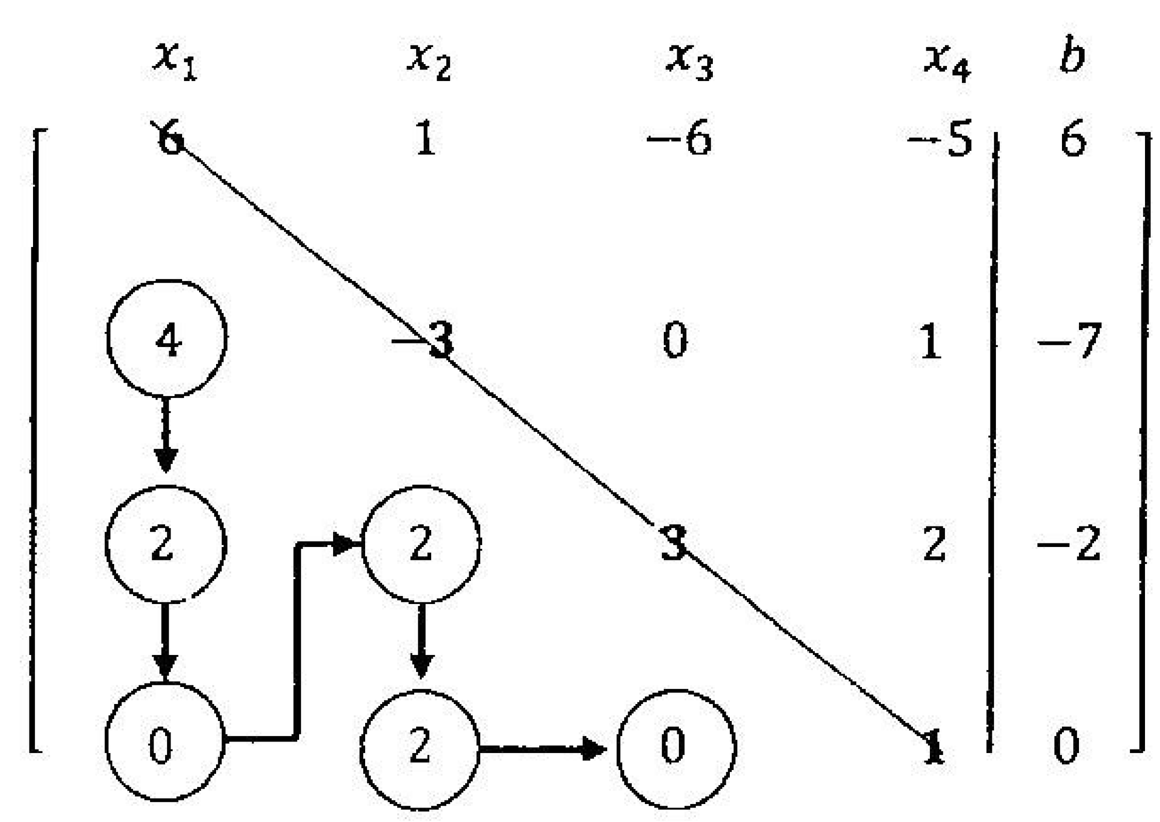
Sub. into Eq. (4): $2x_2 + x_4 = 0$

Then the maximum absolute error is $\epsilon = |6.002 - 6.000| = 0.002$

Also, the maximum relative error will be

$$\epsilon_r = \frac{0.002}{6.000} = 0.033\%$$

The following figure shows the path of zeroing the elements under the major diagonal.



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3. Solution by Iterations (Indirect Method or Iterative Method)

One can rewrite the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

into summation form as follows

$$\sum_{j=1}^{n} a_{ij}\, x_i = b_i, \quad where \quad i = 1, 2, \cdots, n$$

and also the above summation can be written as

$$x_1 = f(x_2, x_3, \cdots, x_n) = \frac{1}{a_{11}}\left[b_1 - \sum_{j=1, j\neq 1}^{n} a_{1j}\, x_j\right],$$

$$x_2 = f(x_1, x_3, \cdots, x_n) = \frac{1}{a_{22}}\left[b_2 - \sum_{j=1, j\neq 2}^{n} a_{2j}\, x_j\right]$$

$$\vdots$$

$$x_n = f(x_1, x_3, \cdots, x_{n-1}) = \frac{1}{a_{nn}}\left[b_n - \sum_{j=1, j\neq n}^{n} a_{nj}\, x_j\right]$$

So one can recall $x = g(x)$ Method to find $x_1, x_2, \dots, x_n$ by guessing starting points for $x_1^o, x_2^o, \cdots, x_n^o$ and trying to find out the new or modified values of $x_1^{new}, x_2^{new}, \cdots, x_n^{new}$ and by repeating this procedure, one can determine the solution of any system.

I. Jacobi Method

The formula of this method is as follows

$$x_i^{k+1} = \frac{1}{a_{ii}}\left\{b_i - \sum_{\substack{j=1 \\ i\neq j}}^{n} a_{ij}\, x_j^k\right\}, \quad i = 1, 2, \cdots, n$$

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Algorithm: Jacobi Method

1. Given a $\mathbf{Ax} = \mathbf{b}$, $A = [a_{ij}]$ an $n \times n$ matrix and $\mathbf{b} = \{b_i\}$ an n vector.
2. Exchange the contents of rows so that the diagonal elements (a_{ii}) have magnitudes as large as possible relative to the magnitude of other coefficients in the same column ($|a_{ii}| > |a_{ij}|, i \neq j$, for each column).
3. If $a_{ii} = 0, i = 1, 2, \cdots, n$ then Stop. "No Unique Solution"
4. Choose starting points $x_i^o, i = 1, 2, \cdots, n$.
5. For $k = 1, 2, 3, \cdots, until\ termination$ do:

$$x_i^{k+1} = \frac{1}{a_{ii}}\left\{b_i - \sum_{\substack{j=1 \\ i \neq j}}^{n} a_{ij}\, x_j^k\right\}, \quad i = 1, 2, \cdots, n$$

Check for termination:

$$\left|\frac{x_i^{k+1} - x_i^k}{x_i^k}\right| \leq \epsilon_r, or \left|x_i^{k+1} - x_i^k\right| \leq \epsilon$$

End

Output $\mathbf{x}$

Example 4.5

Solve the system

$$2x_1 + x_2 + 9x_3 = 12$$
$$8x_1 + x_2 - x_3 = 8$$
$$x_1 - 7x_2 + 2x_3 = -4$$

by using Jacobi Method with $x_1^o = x_2^o = x_3^o = 0.0000$. (Correct to $4D$ and check for accuracy).

Solution

Rearrange the system

$$8x_1 + x_2 - x_3 = 8 \qquad \cdots(1)$$
$$x_1 - 7x_2 + 2x_3 = -4 \qquad \cdots(2)$$
$$2x_1 + x_2 + 9x_3 = 12 \qquad \cdots(3)$$

by using Jacobi's formula the above system can be written as

$$x_1^{k+1} = \frac{1}{8}[8 - (x_2^k - x_3^k)]$$

$$x_2^{k+1} = -\frac{1}{7}[-4 - (x_1^k + 2x_3^k)]$$

$$x_3^{k+1} = \frac{1}{9}[12 - (2x_1^k + x_2^k)]$$

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k	x_1^{k+1}	x_1^{k+1}	x_1^{k+1}
0	1.0000	0.5714	1.3333
1	1.0952	1.0952	1.0476
2	0.9940	1.0272	0.9683
3	0.9926	0.9901	0.9983
4	1.0010	0.9985	1.0027
5	1.0005	1.0009	0.9999
6	0.9999	1.0001	0.9998
7	1.0000	0.9999	1.0000
8	1.0000	1.0000	1.0000
9	1.0000	1.0000	1.0000

The solution is $x_1 = x_2 = x_3 = 1.000$ because $|x_i^9 - x_i^8| = 0, i = 1, 2, 3$ <u>**Ans.**</u>

Check for accuracy

Sub. into Eq. (1): $8x_1 + x_2 - x_3 = 8.0000$ <u>**O.K.**</u>

Sub. into Eq. (2): $x_1 - 7x_2 + 2x_3 = -4.0000$ <u>**O.K.**</u>

Sub. into Eq. (3): $2x_1 + x_2 + 9x_3 = 12.0000$ <u>**O.K.**</u>

Example 4.6

Solve the system

$$5x - 2y + z = 4$$
$$x + 4y - 2z = 3$$
$$x + 2y + 4z = 17$$

by using Jacobi Method with $x^o = y^o = z^o = 4.0000$. (Correct to $4D$ and check for accuracy).

Solution

The is already arranged, so

$$5x - 2y + z = 4 \qquad \cdots(1)$$
$$x + 4y - 2z = 3 \qquad \cdots(2)$$
$$x + 2y + 4z = 17 \qquad \cdots(3)$$

by using Jacobi's formula the above system can be written as

$$x^{k+1} = \frac{1}{5}[4 - (-2y^k + z^k)]$$

$$y^{k+1} = \frac{1}{4}[3 - (x^k - 2z^k)]$$

$$z^{k+1} = \frac{1}{4}[17 - (x^k + 2y^k)]$$

k	x^{k+1}	y^{k+1}	z^{k+1}
0	1.6000	1.7500	1.2500
1	1.2500	0.9750	2.9750
2	0.5950	1.9250	3.4500
⋮	⋮	⋮	⋮
21	1.0000	2.0000	3.0000
22	1.0000	2.0000	3.0000

The solution is $x = 1.0000, y = 2.0000$ *and* $z = 3.000$ because $|x^{22} - x^{21}| = |y^{22} - y^{21}| = |z^{22} - z^{21}| = 0$ **Ans.**

Check for accuracy

Sub. into Eq. (1): $5x - 2y + z = 4.0000$ **O.K.**

Sub. into Eq. (2): $x + 4y - 2z = 3.0000$ **O.K.**

Sub. into Eq. (3): $x + 2y + 4z = 17.0000$ **O.K.**

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II. Gauss-Seidel Method

The Jacobi's formula can be modified by using the improved results in the same iteration step, this improvement can be written in summation notation as

$$x_i^{k+1} = \frac{1}{a_{ii}}\left\{b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{k}\right\}, \quad i = 1, 2, \cdots, n$$

Algorithm: Gauss-Seidel Method

1. Given a $\mathbf{Ax} = \mathbf{b}$, $\mathbf{A} = [a_{ij}]$ an $n \times n$ matrix and $\mathbf{b} = \{b_i\}$ an n vector.
2. Exchange the contents of rows so that the diagonal elements (a_{ii}) have magnitudes as large as possible relative to the magnitude of other coefficients in the same column ($|a_{ii}| > |a_{ij}|, i \neq j$, for each column).
3. If $a_{ii} = 0, i = 1, 2, \cdots, n$ then Stop. "No Unique Solution"
4. Choose starting points $x_i^o, i = 1, 2, \cdots, n$.
5. For $k = 1, 2, 3, \cdots, until\ termination$ do:

   $$x_i^{k+1} = \frac{1}{a_{ii}}\left\{b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{k}\right\}, \quad i = 1, 2, \cdots, n$$

   Check for termination:

   $$\left|\frac{x_i^{k+1} - x_i^k}{x_i^k}\right| \le \epsilon_r, or \left|x_i^{k+1} - x_i^k\right| \le \epsilon$$

   End

   Output **x**

Example 4.7

Solve the system of Example 4.5

$$2x_1 + x_2 + 9x_3 = 12$$
$$8x_1 + x_2 - x_3 = 8$$
$$x_1 - 7x_2 + 2x_3 = -4$$

by using Gauss-Seidel Method with $x_1^o = x_2^o = x_3^o = 0.0000$. (Correct to $4D$ and check for accuracy).

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Solution

Rearrange the system

$$8x_1 + x_2 - x_3 = 8 \quad \cdots(1)$$
$$x_1 - 7x_2 + 2x_3 = -4 \quad \cdots(2)$$
$$2x_1 + x_2 + 9x_3 = 12 \quad \cdots(3)$$

by using Jacobi's formula the above system can be written as

$$x_1^{k+1} = \frac{1}{8}[8 - (x_2^k - x_3^k)]$$

$$x_2^{k+1} = -\frac{1}{7}[-4 - (x_1^{k+1} + 2x_3^k)]$$

$$x_3^{k+1} = \frac{1}{9}[12 - (2x_1^{k+1} + x_2^{k+1})]$$

k	x_1^{k+1}	x_1^{k+1}	x_1^{k+1}
0	1.0000	0.7143	1.0317
1	1.0397	1.0147	0.9895
2	0.9969	0.9966	1.0011
3	1.0006	1.0004	0.9998
4	0.9999	0.9999	1.0000
5	1.0000	1.0000	1.0000
6	1.0000	1.0000	1.0000

The solution is $x_1 = x_2 = x_3 = 1.000$ because $|x_i^9 - x_i^8| = 0, i = 1, 2, 3$ **Ans.**

Check for accuracy

Sub. into Eq. (1): $8x_1 + x_2 - x_3 = 8.0000$ **O.K.**

Sub. into Eq. (2): $x_1 - 7x_2 + 2x_3 = -4.0000$ **O.K.**

Sub. into Eq. (3): $2x_1 + x_2 + 9x_3 = 12.0000$ **O.K.**

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Example 4.8

Solve the system of Example 4.6

$$5x - 2y + z = 4$$
$$x + 4y - 2z = 3$$
$$x + 2y + 4z = 17$$

by using Gauss-Seidel Method with $x^o = y^o = z^o = 4.0000$. (Correct to $4D$ and check for accuracy).

Solution

The is already arranged, so

$$5x - 2y + z = 4 \quad \cdots(1)$$
$$x + 4y - 2z = 3 \quad \cdots(2)$$
$$x + 2y + 4z = 17 \quad \cdots(3)$$

by using Jacobi's formula the above system can be written as

$$x^{k+1} = \frac{1}{5}[4 - (-2y^k + z^k)]$$

$$y^{k+1} = \frac{1}{4}[3 - (x^{k+1} - 2z^k)]$$

$$z^{k+1} = \frac{1}{4}[17 - (x^{k+1} + 2y^{k+1})]$$

k	x^{k+1}	y^{k+1}	z^{k+1}
0	1.6000	2.3500	2.6750
1	1.2050	1.7863	3.0556
2	0.9634	2.520	2.9982
⋮	⋮	⋮	⋮
9	1.0000	2.0000	3.0000
10	1.0000	2.0000	3.0000

The solution is $x = 1.0000, y = 2.0000$ *and* $z = 3.000$ because $|x^{10} - x^9| = |y^{10} - y^9| = |z^{10} - z^9| = 0$ **Ans.**

Check for accuracy

Sub. into Eq. (1): $5x - 2y + z = 4.0000$ **O.K.**

Sub. into Eq. (2): $x + 4y - 2z = 3.0000$ **O.K.**

Sub. into Eq. (3): $x + 2y + 4z = 17.0000$ **O.K.**

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SYSTEMS OF LINEAR EQUATIONS

SUMMARY: SOLUTION OF LINEAR SYSTEMS

Method	Formulas and Procedure
1. Graphical Method	-
2. Gaussian Elimination Method	1. Rearrange the System 2. Perform Elimination Process For $j = k+1, \cdots, n$, do: $q_{jk} = \frac{a_{jk}}{a_{kk}}, b_j = b_j - q_{ik} \times b_k$ For $p = 1, \cdots, n$, do: $a_{jp} = a_{jp} - q_{jk}\, a_{kp}$ End End 3. Start with Back-Substitution $x_n = \frac{a_{n,n+1}}{a_{nn}}$ For $i = n-1, \cdots, 1$, do: $x_i = \frac{1}{a_{ii}}\left(a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j\right)$ End 4. Check for accuracy of the results
3. Jacobi Method	1. Rearrange the System 2. Perform the iteration until termination $x_i^{k+1} = \frac{1}{a_{ii}}\left\{b_i - \sum_{\substack{j=1 \\ i \neq j}}^{n} a_{ij}\, x_j^k\right\}, \quad i = 1, 2, \cdots, n$ 3. Check for accuracy of the results
4. Gauss-Seidel Method	1. Rearrange the System 2. Perform the iteration until termination $x_i^{k+1} = \frac{1}{a_{ii}}\{b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}\, x_j^k\}, \quad i = 1, 2, \cdots, n$ 3. Check for accuracy of the results

5. `inv` MATLAB command	1. Solve the system
	`x=inv(A)*b`
	2. Check the accuracy of the results
	`A*x-b`

Termination Criteria

i. If $\left|x_i^{k+1} - x_i^k\right| \leq \epsilon$ or $\left|\frac{x_i^{k+1} - x_i^k}{x_i^k}\right| \leq \epsilon_r$ ($\epsilon > 0$ *and* $\epsilon_r > 0$, specified tolerances)

ii. After N steps (N, fixed)

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HOME WORKS: SOLUTION OF LINEAR SYSTEMS

GAUSSIAN ELIMINATION METHOD

H.W 4.1

Solve the system

$$x_1 + 3x_2 + 2x_3 = 5$$
$$2x_1 + 4x_2 - 6x_3 = -4$$
$$x_1 + 5x_2 + 3x_3 = 10$$

by using Gaussian Elimination Method. (Correct to $4D$ and check for accuracy).

Answer: $x_1 = -3, x_2 = 2 \text{ and } x_3 = 1$

H.W 4.2

Solve the system

$$x + 5y - 5z - 3w = 18$$
$$2x + 4y - 4z = 12$$
$$x + 4y - 2z + 2w = 10$$
$$2x + 3y + z + 3w = 8$$

by using Gaussian Elimination Method. (Correct to $4D$ and check for accuracy).

Answer: $x = 0.6667(2/3), y = 3.3333\ (10/3), z = 0.6667\ (2/3) \text{ and } w = -1.3333(-4/3)$

H.W 4.3

Solve the system

$$x + 4y + 7z + 2w = 10$$
$$4x + 8y + 4z = 8$$
$$x + 3y - 2w = 10$$
$$x + 5y + 4z - 3w = -4$$

by using Gaussian Elimination Method. (Correct to $4D$ and check for accuracy).

Answer: $x = -39, y = 27, z = -13 \text{ and } w = 16$

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THE JACOBI METHOD

H.W 4.4

Solve the system

$$x + 4y + 7z = 10$$
$$4x + 8y + 4z = 8$$
$$10x + 3y - 2z = 10$$

by using Jacobi Method with $x^o = y^o = z^o = 0.0000$. (Correct to $4D$ and check for accuracy).

Answer (Approximate): $x = 1.4359, y = -0.4615$ and $z = 1.4872$

Answer (Closed-Form): $x = 56/39, y = -6/13$ and $z = 58/39$

H.W 4.5

Solve the system

$$2x + 8y - z = 11$$
$$5x - y + z = 10$$
$$-x + y + 4z = 3$$

by using Jacobi Method with $x^o = y^o = z^o = 0.0000$. (Correct to $4D$ and check for accuracy).

Answer: $x = 2, y = 1$ and $z = 1$

H.W 4.6

Solve the system

$$x - 5y - z = -8$$
$$4x + y - z = 13$$
$$2x - y - 6z = -2$$

by using Jacobi Method with $x^o = y^o = z^o = 5.0000$. (Correct to $4D$ and check for accuracy).

Answer: $x = 3, y = 2$ and $z = 1$

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GAUSS-SEIDEL METHOD

H.W 4.7

Solve the system

$$2.5x + 8.1y - 2z = 11.5$$
$$2x + 4.5y - 21z = 3.6$$
$$10x + 2.6y + 4.2z = 17.4$$

by using Gauss-Seidel Method with $x^o = 10000, y^o = 20000 \ and \ z^o = 3.0000$. (Correct to $4D$ and check for accuracy).

Answer: $x = 1.3941, y = 1.0347 \ and \ z = 0.1831$

H.W 4.8

Solve the system of Example 4.6

$$150x - 2y + 4z = 300$$
$$3x + 800y - 2z = 150$$
$$-2x + 2y + 400z = 40$$

by using Gauss-Seidel Method with $x^o = -5.0000, \ y^o = 10.0000 \ and \ z^o = 0.0000$. (Correct to $4D$ and check for accuracy).

Answer: $x = 1.9995, y = 0.1803 \ and \ z = 0.1091$

H.W 4.9

Solve the system of Example 4.6

$$4x - 2y - z = 40$$
$$x - 6y + 2z = -28$$
$$x - 2y + 12z = -86$$

by using Gauss-Seidel Method with $x^o = y^o = z^o = 3.0000$. (Correct to $4D$ and check for accuracy).

Answer (Approximate): $x = 10.1094 , y = 3.8984 \ and \ z = -7.3594$

Answer (Closed-Form): $x = 647/64, y = 499/128 \ and \ z = -471/64$

Chapter 3

SYSTEMS OF LINEAR EQUATIONS

SOLUTION OF LINEAR SYSTEMS: APPLICATIONS IN CIVIL ENGINEERING

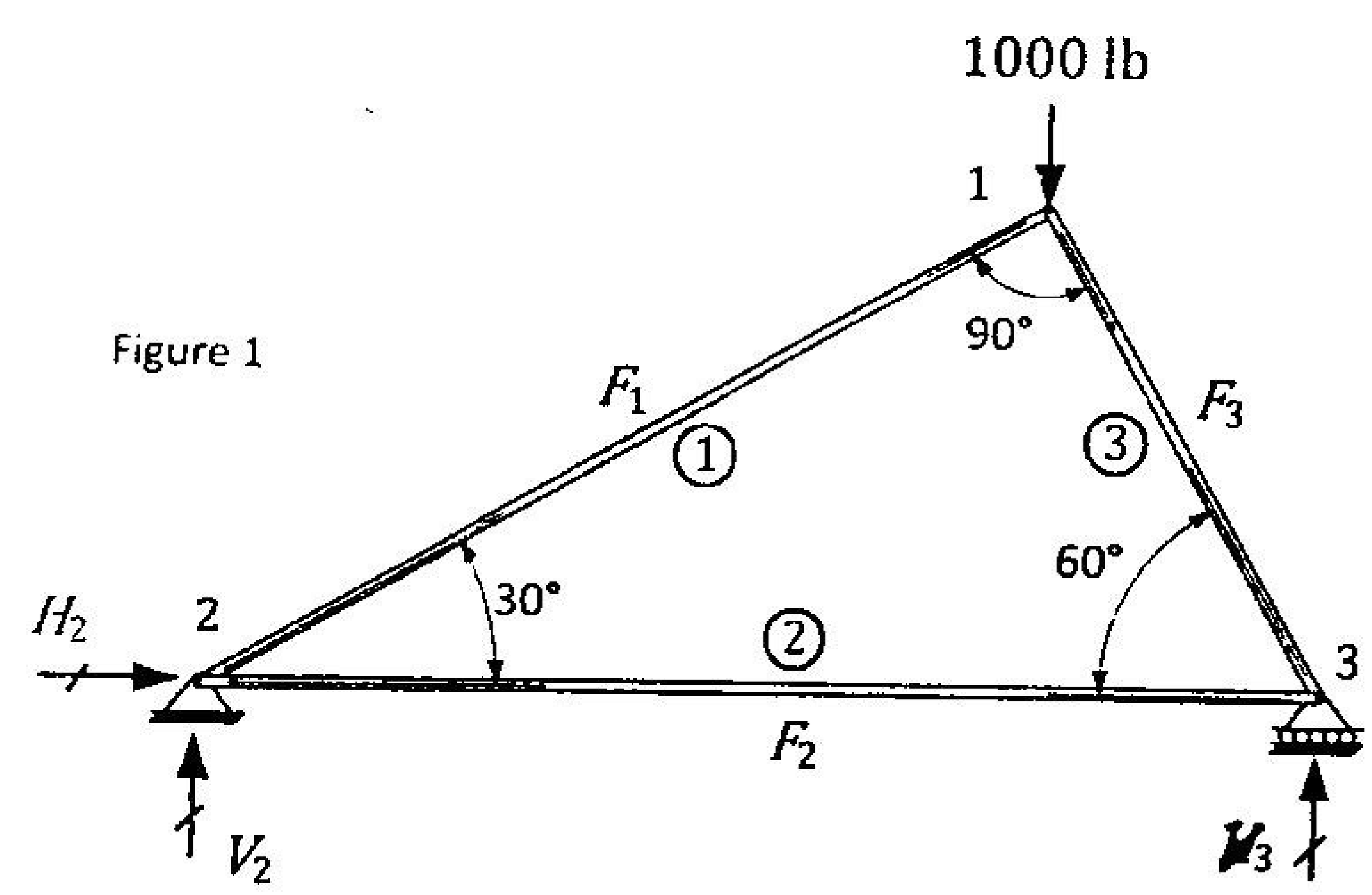
(ANALYSIS OF STATICALLY DETERMINATE TRUSSES)

Case Study 4.1: (Illustrative Case Study)

Figure 1 shows an example of three-member truss. The forces (F) represent either tension or compression on the members of the truss. External reactions (H_2, V_2 and V_3) are forces which characterize how the truss interacts with the supporting surface. It is observed that the effect of the external loading of 1000 lb is distributed among the various members of the truss.

This type of structure can be described as s system of coupled linear algebraic equations. Free-body force diagrams are shown for each node in Figure 2. The sum of the forces in both horizontal and vertical directions must be zero at each node, because the system is in equilibrium. Therefore,

Figure 1



For Node No. 1

$$\sum F_H = 0 = -F_1 \cos 30^o + F_3 \cos 60^o + F_{1,h}$$

$$\sum F_V = 0 = -F_1 \sin 30^o - F_3 \sin 60^o + F_{1,v}$$

For Node No. 2

$$\sum F_H = 0 = F_2 + F_1 \cos 30^o + F_{2,h} + H_2$$

$$\sum F_V = 0 = F_1 \cos 30^o + F_{2,v} + V_2$$

For Node No. 2

$$\sum F_H = 0 = -F_2 - F_3 \cos 60^o + F_{3,h}$$

$$\sum F_V = 0 = F_3 \sin 60^o + F_{3,v} + V_3$$

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Where $F_{i,h}$ is the external horizontal force applied to node i (where the positive force is from left to right) and $F_{i,v}$ is the external vertical force applied to node i (where the positive force is upward). Thus, in this case study , the 1000 lb downward force on node 1 corresponds to $F_{1,v} = -1000$. For this case all other $F_{i,v}$ and $F_{i,h}$'s are zero.

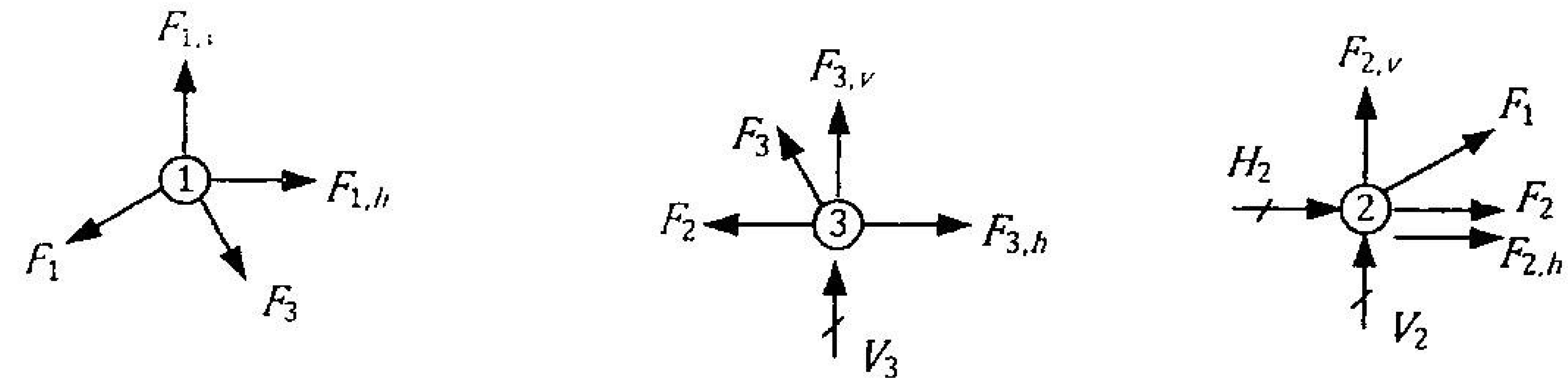


Figure 2

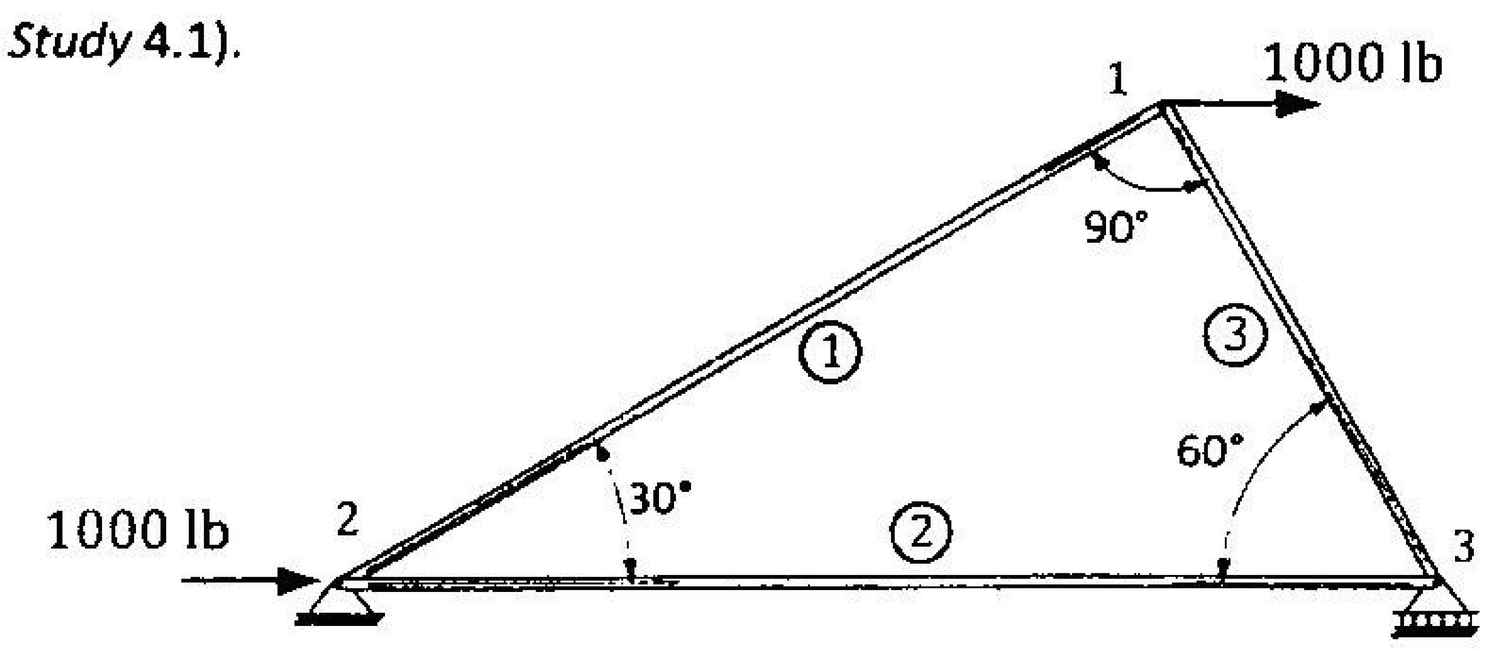
$$\begin{bmatrix} 0.866 & 0 & -0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.866 & 0 & 0 & 0 \\ -0.866 & -1 & 0 & -1 & 0 & 0 \\ -0.5 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -0.866 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ H_2 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solve the above system to find the unknown vector (F_1, F_2, F_3, H_2, V_2 and V_3).

Answer: $F_1 = -500\ lb, F_2 = 433 lb, F_3 = -866\ lb, H_2 = 0, V_2 = 250\ lb$ and $V_3 = 750\ lb$

Case Study 4.2:

Determine the forces in members 1, 2 and 3, and the reactions at nodes 2 and 3. (*Note: Perform the same computation as in Case Study* 4.1).



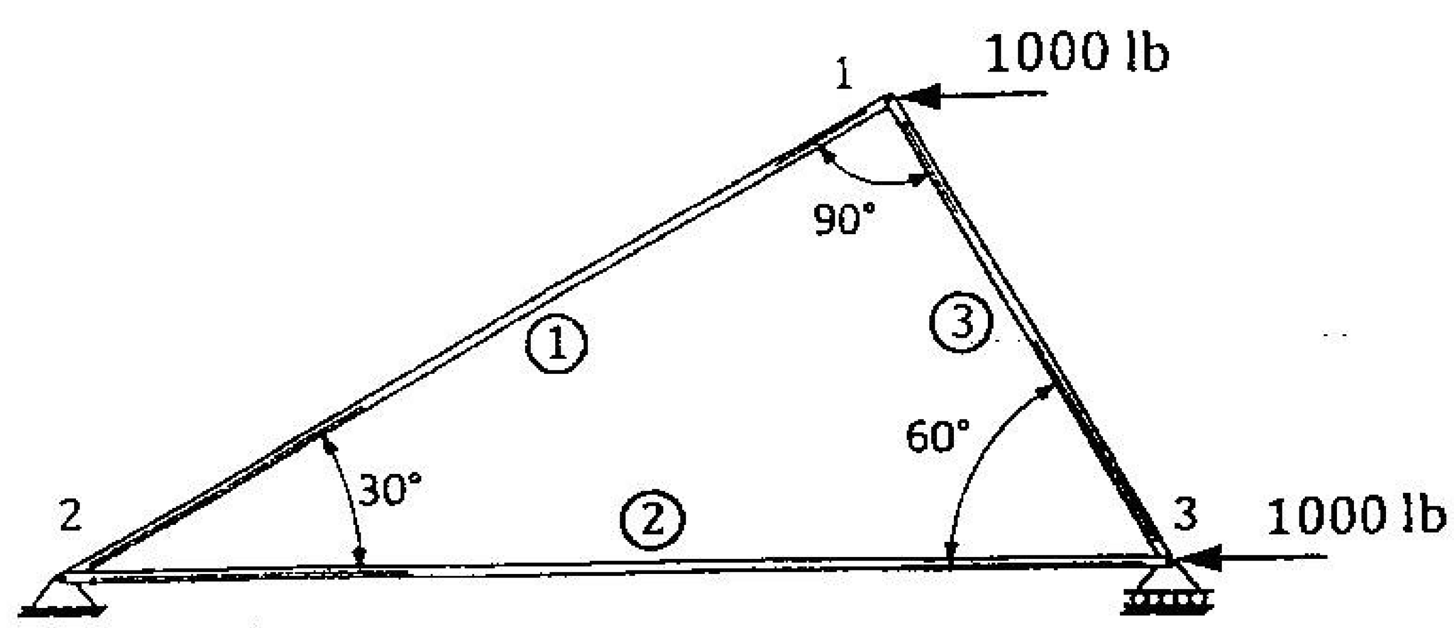
Answer: $F_1 = 866\ lb, F_2 = 250\ lb, F_3 = -500\ lb, H_2 = -2000\ lb, V_2 = -433\ lb$ and $V_3 = 433\ lb$

Chapter 3

SYSTEMS OF LINEAR EQUATIONS

Case Study 4.3:

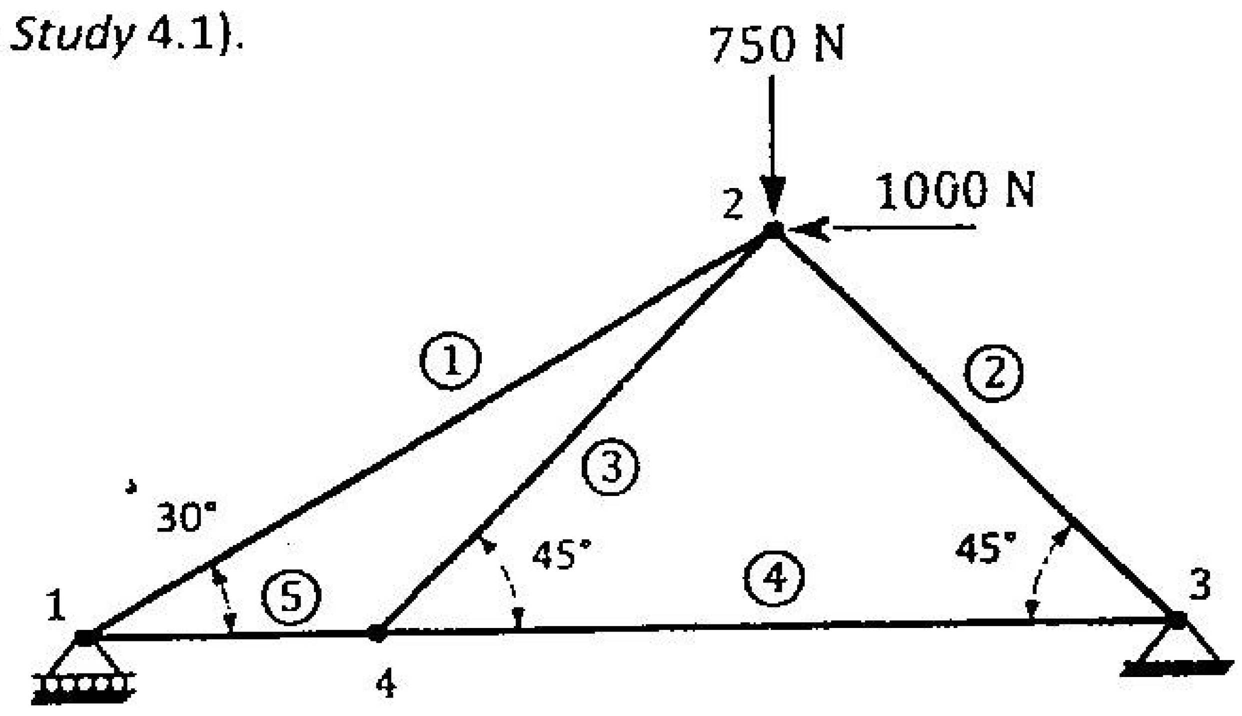
Determine the forces in members 1, 2 and 3, and the reactions at nodes 2 and 3. (*Note: Perform the same computation as in Case Study 4.1*).



Answer: $F_1 = -866\ lb, F_2 = -1250\ lb, F_3 = 500\ lb, H_2 = 2000\ lb, V_2 = 433\ lb$ and $V_3 = -433\ lb$

Case Study 4.4:

Determine the forces in members 1, 2, 3, 4 and 5, and the reactions at nodes 1 and 3. (*Note: Perform the same computation as in Case Study* 4.1).



Case Study 4.5:

Determine the forces in members 1 to 7, and the reactions at nodes 1 and 4. (*Note: Perform the same computation as in Case Study* 4.1).