# Almustafa University College

# Civil Engineering Department

## Mathematic II

## Second Stage

## Multiple Integral

# Multiple Integrals

## Double Integrals

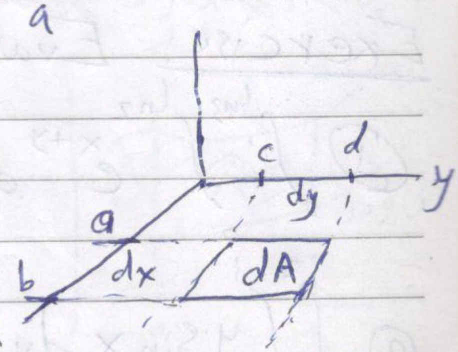
Double Integrals :- Let $R$ be the rectangle defined by the inequalities $a \leq x \leq b$ , $c \leq y \leq d$

If $f(x,y)$ is continuous on this rectangle, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

if $f(x,y)$ is nonnegative on $R$

the double integral $\iint_R f(x,y)\,dA$,




represents the volume of the solid $S$ bounded above by the surface $z = f(x,y)$ and below by the Rectangle $R$.

$$\therefore \text{Vol}(S) = \int_a^b \int_c^d f(x,y)\,dA$$

**Example①:-** Evaluate $\iint_R y^2 x\,dA$ over the rectangle $R$:

$$R : \{(x,y) :- -3 \leq x \leq 2 \ , \ 0 \leq y \leq 1\}.$$

**Solution:-**

$$\int_{-3}^{2}\int_{0}^{1} y^2 x\,dy\,dx = \int_{-3}^{2} \frac{1}{3} y^3 x \Big]_{y=0}^{y=1} dx = \int_{-3}^{2} \frac{1}{3} x\,dx = \frac{x^2}{6}\Big]_{x=-3}^{x=2}$$

$$= \frac{-5}{6}$$

**Example ②:** Use a double integral to find the volume of the solid that is bounded above by the plane $Z = 4 - x - y$ and below by the rectangle $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 2\}$

**Solution**

$$V = \iint_R (4 - x - y)\, dA = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$$

$$= \int_0^2 4x - \frac{x^2}{2} - yx \Big]_0^1 dy$$

$$= \int_0^2 \left(\frac{7}{2} - y\right) dy = \frac{7}{2}y - \frac{y^2}{2}\Big]_0^2 = 5$$

**Exercise:** Evaluate the integrals

① $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y}\, dy\, dx$

② $\int_0^2 \int_0^1 y \sin x\, dy\, dx$

③ $\int_0^3 \int_0^1 x (x^2 + y)^{\frac{1}{2}}\, dx\, dy$

④ $\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\, dy\, dx$

⑤ $\int_{\pi/2}^{\pi} \int_1^2 x \cos xy\, dy\, dx$

⑥ $\int_3^4 \int_1^2 \frac{1}{(x+y)^2}\, dy\, dx$

⑦ $\int_0^3 \int_0^4 \sqrt{25 - x^2 - y^2}\, dy\, dx$

⑧ $\int_0^1 \int_{-1}^1 \sqrt{4 - x^2}\, dy\, dx$

⑨ The Volume under the surface $Z = 3x^3 + 3x^2 y$ and over the rectangle $R: 1 \le x \le 3,\ 0 \le y \le 2$

(10) $\displaystyle\iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\, dA$ , $R: 0 \leq x \leq 1$ , $0 \leq y \leq 1$

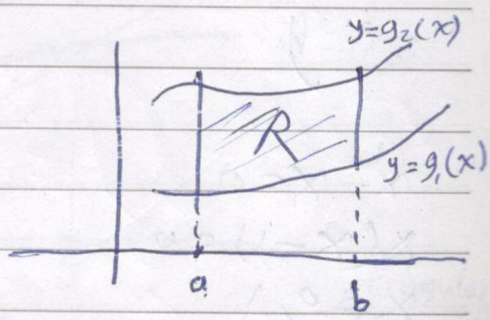
(11) $\displaystyle\iint_R (x\sin y - y\sin x)\, dA$, $R: 0 \leq x \leq \frac{\pi}{2}$ , $0 \leq y \leq \frac{\pi}{3}$

(12) $\displaystyle\iint_R \cos(x+y)\, dA$ , $R: \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$ , $0 \leq y \leq \frac{\pi}{2}$

Theorem (a) IF $R$ is defined by $a \leq x \leq b$ ; $g_1(x) \leq y \leq g_2(x)$

then $\displaystyle\iint_R f(x,y)\, dA$

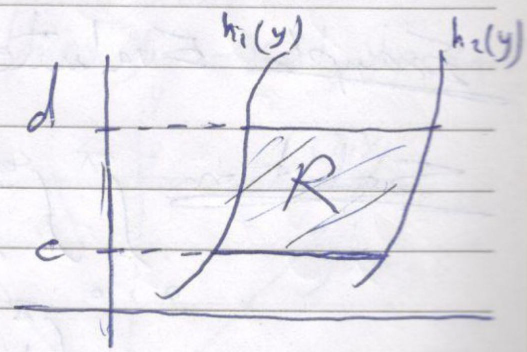
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$




(b) IF $R$ is defined by $c \leq y \leq d$ , $h_1(y) \leq x \leq h_2(y)$

then $\displaystyle\iint_R f(x,y)\, dA$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$



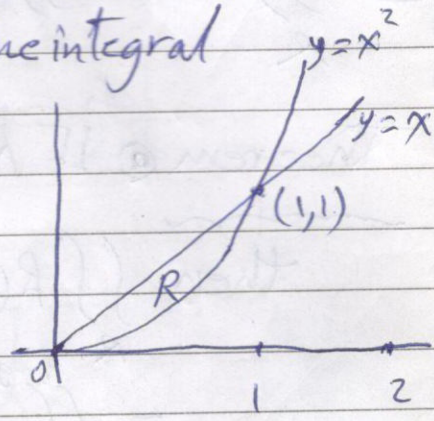
Example Evaluate $\int_0^1 \int_{x^2}^{x} y^2 x \, dy \, dx$

Solution :- $= \int_0^1 \frac{y^3}{3} x \Big]_{x^2}^{x} dx = \int_0^1 \left(\frac{x^4}{3} - \frac{x^7}{3}\right) dx$

$= \frac{x^5}{15} - \frac{x^8}{24} \Big]_0^1 =$

Now to reverse the order of the integral

$\int_0^1 \int_y^{\sqrt{y}} y^2 x \, dx \, dy$




$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

$\therefore y = 0, 1$

$y = x \Rightarrow x = y$

$y = x^2 \Rightarrow x = \sqrt{y}$

~~$x^2$~~

Example Evaluate $\int_0^{\pi} \int_0^{\cos y} x \sin y \, dx \, dy$

Solution $\int_0^{\pi} \int_0^{\cos y} x \sin y \, dx \, dy = \int_0^{\pi} \frac{x^2}{2} \sin y \Big]_0^{\cos y} dy$

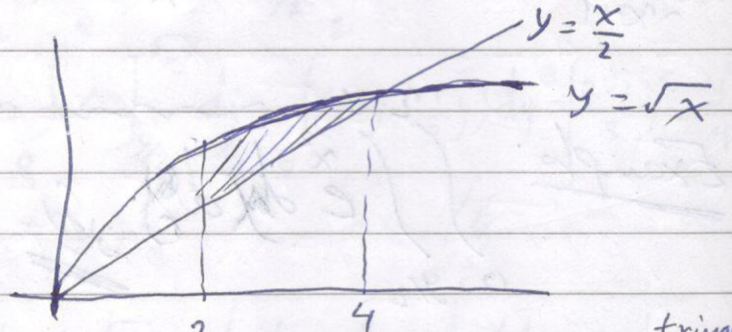
$= \int_0^{\pi} \frac{1}{2} \cos^2 y \sin y \, dy = \left[\frac{-1}{6} \cos^3 y\right]_0^{\pi} = \frac{1}{3}$

**Example:-** Evaluate $\iint\limits_R xy\,dA$ over the region R enclosed between $y=\frac{1}{2}x$, $y=\sqrt{x}$, $x=2$ and $x=4$

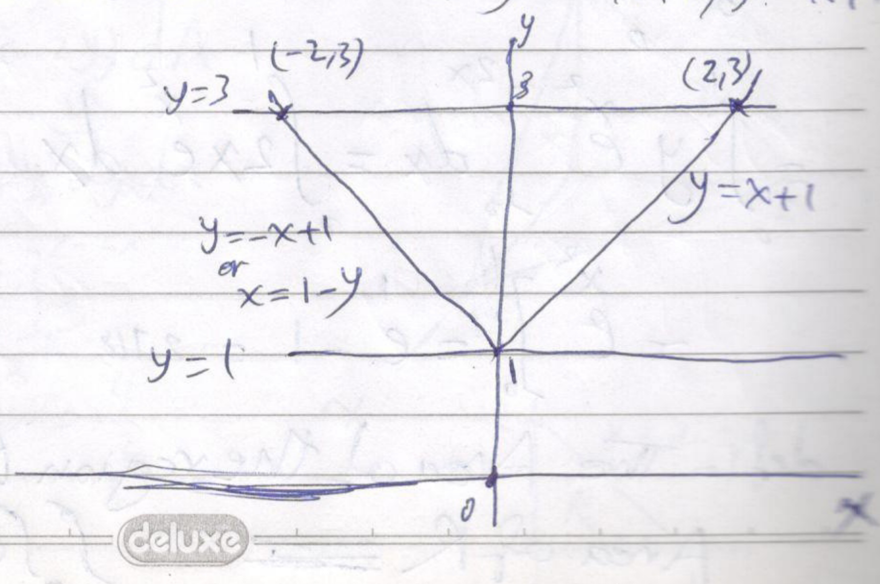
$$\iint\limits_R xy\,dA = \int_2^4\int_{\frac{x}{2}}^{\sqrt{x}} xy\,dy\,dx = \int_2^4 \left.\frac{xy^2}{2}\right]_{\frac{x}{2}}^{\sqrt{x}} dx$$

$$= \int_2^4 \left(\frac{x^2}{2}-\frac{x^3}{8}\right)dx = \left[\frac{x^3}{6}-\frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6}-\frac{256}{32}\right)-\left(\frac{8}{6}-\frac{16}{32}\right)$$

$$= \frac{11}{6}$$




**Example:-** Evaluate $\iint\limits_R (2x-y^2)\,dA$, over the tringular region R enclosed between the lines $y=-x+1$, $y=x+1$ and $y=3$

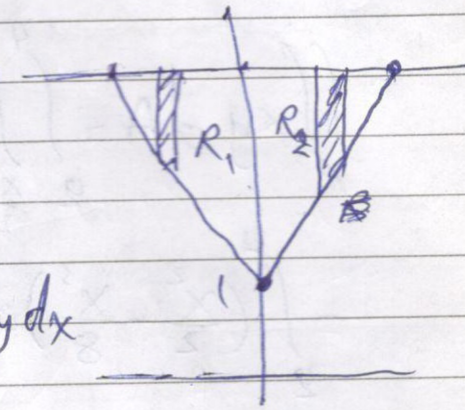


Solution: $\iint\limits_R (2x-y^2)\,dA = \int_1^3\int_{1-y}^{y-1}(2x-y^2)\,dx\,dy$

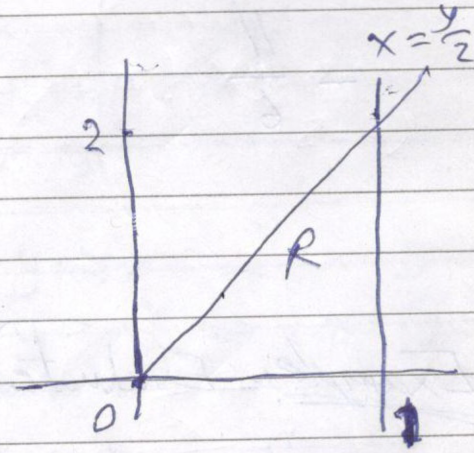
$\iint\limits_R (2x-y^2)\,dA$

$= \iint\limits_{R_1}(2x-y^2)\,dA + \iint\limits_{R_2}(2x-y^2)\,dA$

$= \int_{-2}^{0}\int_{-x+1}^{3}(2x-y^2)\,dy\,dx + \int_{0}^{2}\int_{x+1}^{3}(2x-y^2)\,dy\,dx$




Example $\int_0^2\int_{y/2}^{1} e^{x^2}\,dx\,dy$




$= \int_0^1\int_0^{2x} e^{x^2}\,dy\,dx$

$x=\frac{y}{2}$

$= \int_0^1 y\,e^{x^2}\Big]_0^{2x} dx = \int_0^1 2x\,e^{x^2}\,dx$ $\quad \therefore y = 2x$

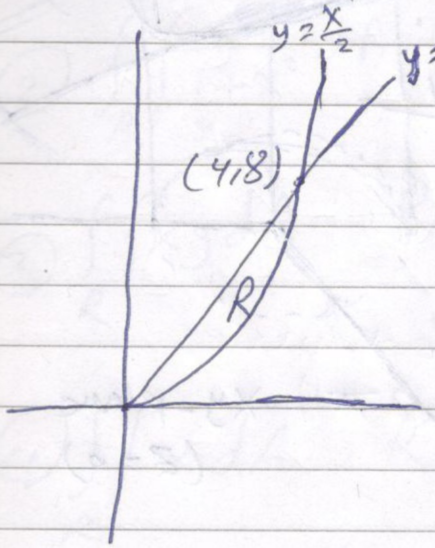
$= e^{x^2}\Big]_0^1 = e^1 - 1 = 2.718 - 1 =$

def:- The Area of the region R by

Area of R $= \iint\limits_R dA$

**Example** Use a double integral to find the area of the region $R$ enclosed between the parabola $y=\frac{1}{2}x^2$ and the line $y=2x$.

**Solution** area of $R$ $=\iint\limits_R dA = \int_0^4\int_{\frac{x^2}{2}}^{2x} dy\,dx = \int_0^4 y\Big|_{\frac{x^2}{2}}^{2x} dx$




$$=\int_0^4\left(2x-\frac{1}{2}x^2\right)dx = x^2-\frac{x^3}{6}\Big|_0^4 = \frac{16}{3}$$

$$\text{or } \iint\limits_R dA = \int_0^8\int_{y/2}^{\sqrt{2y}} dx\,dy$$

$$=\int_0^8 x\Big]_{y/2}^{\sqrt{2y}} dy = \int_0^8\left(\sqrt{2y}-\frac{1}{2}y\right)dy = \left[\frac{2\sqrt{2}}{3}y^{3/2}-\frac{y^2}{4}\right]_0^8$$

$$=\frac{16}{3}$$

**Example:** Use a double integral to find the volume of the solid bounded by the coordinate plane and the plane $Z=4-4x-2y$

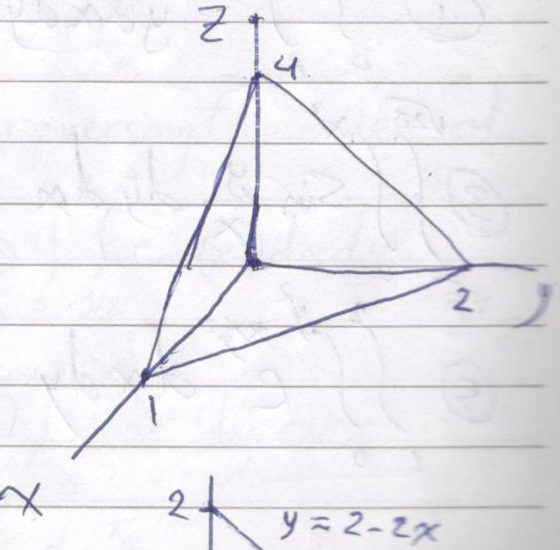
**Solution**

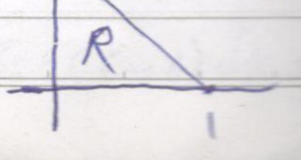
$$V=\iint\limits_R (4-4x-2y)\,dA$$

$$=\int_0^1\int_0^{(2-2x)}(4-4x-2y)\,dy\,dx$$

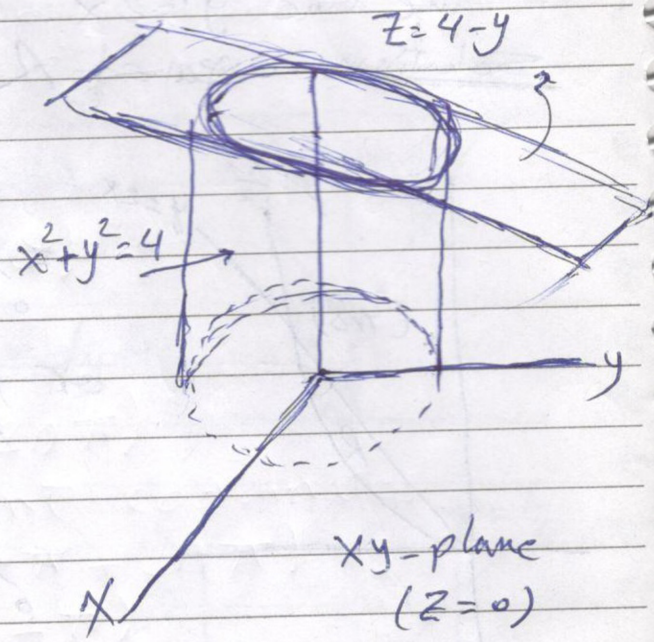
$$=\int_0^1 4y-4xy-y^2\Big]_0^{2-2x}$$

$$=\int_0^1(4-8x+4x^2)\,dx = \frac{4}{3}$$



**Example:-** Find the volume of the solid bounded by cylinder $x^2+y^2=4$ and the planes $y+Z=4$ and $Z=0$




**Solution**

$$V=\iint_R (4-y)\,dA$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx$$

$$=\int_{-2}^{2}\Big[4y-\frac{1}{2}y^2\Big]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$

$$=\int_{-2}^{2} 8\sqrt{4-x^2}\,dx=8(2\pi)=16\pi$$

**Exercises** Evaluate

① $\displaystyle\int_0^3\int_0^{\sqrt{9-y^2}} y\,dx\,dy$

② $\displaystyle\int_{1/4}^{1}\int_{x^2}^{x}\sqrt{\frac{x}{y}}\,dy\,dx$

③ $\displaystyle\int_0^{\sqrt{2\pi}}\int^{x^3} \sin\frac{y}{x}\,dy\,dx$

④ $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\sin y} e^{x}\cos y\,dx\,dy$

⑤ $\displaystyle\int_1^{\sqrt{\pi}}\int_0^{y^2} e^{x/y^2}\,dx\,dy$

⑥ $\displaystyle\int_0^1\int_0^1 y\sqrt{x^2-y^2}\,dy\,dx$

Example Evaluate the double integral

① $\iint_R x\cos xy\,dA$ ; R is the region enclosed by $x=1$, $x=2$, $y=\pi$, and $y=2\pi/x$.

② $\iint_R \frac{1}{1+x^2}\,dA$ ; R is the triangular region with vertices $(0,0)$, $(1,1)$ and $(0,1)$

③ $\iint_R xy\,dA$ ; R is the region enclosed by $y=\sqrt{x}$, $y=6-x$, and $y=0$

④ $\iint_R x\,dA$ ; R is the region enclosed by $y=\sin^{-1}x$, $x=1/\sqrt{2}$ and $y=0$

⑤ Find the volume of the solid bounded above by the paraboloid $z=x^2+3y^2$, below by the plane $z=0$ and laterally by $y=x^2$ and $y=x$

Example

Evaluate the integral by first reversing the order of integration.

① $\int_0^1\int_{4x}^4 e^{-y^2}\,dy\,dx$

② $\int_0^2\int_{y/2}^1 \cos(x^2)\,dx\,dy$

③ $\int_0^4\int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy$

④ $\int_1^3\int_0^{\ln x} x\,dy\,dx$

⑤ $\int_0^1\int_0^{\cos^{-1}x} x\,dy\,dx$

⑥ $\int_0^1\int_{\sin^{-1}y}^{\pi/2} \sec^2(\cos x)\,dx\,dy$